Solutions to Homework Problems on the Green Function Method
Spring 2009, Math 246, Professor David Levermore

1. Compute the Green functions associated with the following differential operators L.

$$\begin{aligned} &\text{a)} \quad \mathrm{L} = \mathrm{D}^2 + 8\mathrm{D} - 20 \\ &\text{b)} \quad \mathrm{L} = \mathrm{D}^2 + 8\mathrm{D} + 20 \\ &\text{c)} \quad \mathrm{L} = \mathrm{D}^4 + 13\mathrm{D}^2 + 36 \end{aligned}$$

Solutions: The final answers are:

$$\begin{aligned} &\text{a)} \quad g(t) = \tfrac{1}{12}\left(e^{2t} - e^{-10t}\right), \\ &\text{b)} \quad g(t) = \tfrac{1}{2}e^{-4t}\sin(2t), \\ &\text{c)} \quad g(t) = \tfrac{1}{15}\sin(3t) - \tfrac{1}{10}\sin(2t). \end{aligned}$$

2. Use the Green function method to find a general solution of the following equations.

$$\begin{aligned} &\text{a)} \quad \mathrm{D}^2 y - \mathrm{D}y - 2y = \frac{1}{e^t + 1} \\ &\text{b)} \quad \mathrm{D}^2 y + y = \tan(t) \\ &\text{c)} \quad \mathrm{D}^2 y + y = \frac{1}{16 + 9\cos(t)^2} \end{aligned}$$

Solutions: The answers expressed first as definite integrals and then explicitly are:

$$\begin{aligned} \text{a)} \quad y &= c_1 e^{2t} + c_2 e^{-t} + \tfrac{1}{3}e^{2t}\int_0^t \frac{e^{-2s}}{e^s + 1}\,\mathrm{d}s - \tfrac{1}{3}e^{-t}\int_0^t \frac{e^s}{e^s + 1}\,\mathrm{d}s \\ &= c_1 e^{2t} + c_2 e^{-t} + \tfrac{1}{3}e^{2t}\left[-\tfrac{1}{2}(1 - e^{-t})^2 - \log\left(\frac{e^{-t} + 1}{2}\right)\right] - \tfrac{1}{3}e^{-t}\log\left(\frac{e^t + 1}{2}\right), \\ \text{b)} \quad y &= c_1\cos(t) + c_2\sin(t) + \sin(t)\int_0^t \cos(s)\tan(s)\,\mathrm{d}s - \cos(t)\int_0^t \sin(s)\tan(s)\,\mathrm{d}s \\ &= c_1\cos(t) + c_2\sin(t) + \sin(t) - \cos(t)\,\tfrac{1}{2}\log\left(\frac{1 + \sin(t)}{1 - \sin(t)}\right), \\ \text{c)} \quad y &= c_1\cos(t) + c_2\sin(t) + \sin(t)\int_0^t \frac{\cos(s)}{16 + 9\cos(s)^2}\,\mathrm{d}s - \cos(t)\int_0^t \frac{\sin(s)}{16 + 9\cos(s)^2}\,\mathrm{d}s \\ &= c_1\cos(t) + c_2\sin(t) + \frac{\sin(t)}{30}\log\left(\frac{5 + 3\sin(t)}{5 - 3\sin(t)}\right) \\ &\qquad + \frac{\cos(t)}{12}\left[\tan^{-1}\left(\tfrac{3}{4}\cos(t)\right) - \tan^{-1}\left(\tfrac{3}{4}\right)\right]. \end{aligned}$$