

First-Order Ordinary Differential Equations II: Autonomous Case

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These notes cover some of the material that we covered in class on first-order ordinary differential equations. As the presentation of this material in class was somewhat different from that in the book, I felt that a written review closely following the class presentation might be appreciated.

3. First-Order Equations: Autonomous Case

3.1. Autonomous Equations. The next simple form of (1.10) to treat is that of autonomous equations. These have the form

$$\frac{dy}{dt} = g(y). \quad (3.1)$$

Here the derivative of y with respect to t is given by a function $g(y)$ that is independent of t .

3.1.1. Recipe for Solving Autonomous Equations. Just as we did for the linear case, we will reduce the autonomous case to the explicit case. The trick to doing this is to consider t to be a function of y . This trick works over intervals over which the solution $y(t)$ is a strictly monotonic function of t . This will be the case over intervals where $g(y(t))$ is never zero. In that case, by the chain rule we have

$$\frac{dt}{dy} = \frac{1}{\frac{dy}{dt}} = \frac{1}{g(y)}.$$

This is an explicit equation for the derivative of t with respect to y . It can be integrated to obtain

$$t = \int \frac{dy}{g(y)} = G(y) + c, \quad \text{where } G'(y) = \frac{1}{g(y)} \text{ and } c \text{ is any constant.} \quad (3.2)$$

We claim that if you solve this equation for y as a differentiable function of t then the result will be a solution of (3.1) wherever $g(y) \neq 0$.

Indeed, suppose that $Y(t)$ is differentiable and satisfies

$$t = G(Y(t)) + c.$$

Upon differentiating both sides of this equation with respect to t we see that

$$1 = G'(Y(t)) \frac{dY(t)}{dt} = \frac{1}{g(Y(t))} \frac{dY(t)}{dt}, \quad \text{wherever } g(Y(t)) \neq 0.$$

It follows that $y = Y(t)$ satisfies (3.1).

Notice that (3.2) gives solutions of (3.1) implicitly as

$$t = G(y) + c, \quad \text{where } c \text{ is an arbitrary constant.}$$

To find an explicit solution, you must solve this equation for y as a function of t . This means finding an inverse function of G — namely, a function G^{-1} with property that

$$G^{-1}(G(y)) = y \quad \text{for every } y \text{ in some subset of the domain of } G.$$

For every such an inverse function, a family of explicit solutions to (3.1) is then given by

$$y = G^{-1}(t - c). \quad (3.3)$$

As we will see in examples below, such a solution may not exist for every value of $t - c$, or there may be more than one solution.

3.1.2. Initial-Value Problems for Autonomous Equations. In order to pick a unique solution from the family (3.3) one must impose an additional condition that determines c . As for linear equations, we do this by imposing an *initial condition* of the form $y(t_I) = y_I$, where t_I is called the *initial time* and y_I is called the *initial value*. The combination of the differential equation (3.1) with the above initial condition is the *initial-value problem*

$$\frac{dy}{dt} = g(y), \quad y(t_I) = y_I. \quad (3.4)$$

There are two possibilities: either $g(y_I) = 0$ or $g(y_I) \neq 0$.

If $g(y_I) = 0$ then it is clear that $y(t) = y_I$ is a solution of (3.4) that is defined for every t . Because this solution does not depend on t it is called a *stationary solution*. Every zero of g is also called a *stationary point* because it yields such a stationary solution. These points are also sometimes called either *equilibrium points* or *critical points*.

Example. Consider the autonomous equation

$$\frac{dy}{dt} = 4y - y^3.$$

Because $4y - y^3 = y(2 - y)(2 + y)$, we see that $y = 0$, $y = 2$, and $y = -2$ are stationary points of this equation.

On the other hand, if $g(y_I) \neq 0$ then you can try to use recipe (3.2) to obtain an implicit relation $t = G(y) + c$. The initial condition then implies that

$$t_I = G(y_I) + c,$$

whereby $c = t_I - G(y_I)$. The solution of the initial-value problem (3.4) is then implicitly given by

$$t = G(y) + t_I - G(y_I).$$

To find an explicit solution, you must solve this equation for y as a function of t . There may be more than one solution of this equation. If so, be sure to take the one that satisfies the initial condition. There will always be exactly solution that does satisfy the initial condition for all times t in some open interval that contains the initial time t_I . The largest such interval is the interval of existence for the solution.

Example. Find the solution to the initial-value problem

$$\frac{dy}{dt} = y^2, \quad y(0) = y_o.$$

Identify its interval of existence.

Solution. We see that $y = 0$ is the only stationary point of this equation. Hence, if $y_o = 0$ then $y(t) = 0$ is a stationary solution whose interval of existence is $(-\infty, \infty)$. So let us suppose that $y_o \neq 0$. The solution is given implicitly by

$$t = \int \frac{dy}{y^2} = -\frac{1}{y} + c.$$

The initial condition then implies that

$$0 = -\frac{1}{y_o} + c,$$

whereby $c = 1/y_o$. The solution is therefore given implicitly by

$$t = -\frac{1}{y} + \frac{1}{y_o}.$$

This may be solved for y explicitly by first noticing that

$$\frac{1}{y} = \frac{1}{y_o} - t = \frac{1 - y_o t}{y_o},$$

and then taking the reciprocal of each side to obtain

$$y = \frac{y_o}{1 - y_o t}.$$

It is clear from this formula that the solution ceases to exist when $t = 1/y_o$. Therefore if $y_o > 0$ then the interval of existence of the solution is $(-\infty, 1/y_o)$ while if $y_o < 0$ then the interval of existence of the solution is $(1/y_o, \infty)$. Notice that both of these intervals contain the initial time $t = 0$. Finally, notice that our explicit solution recovers the stationary solution when $y_o = 0$ even though it was derived assuming that $y_o \neq 0$.

Remark. The above example already shows some important differences between nonlinear and linear equations. For one, it shows that for nonlinear equations the interval of existence can depend on the initial data. This is in marked contrast with linear equations where you can read off the interval of existence from the coefficient and the forcing. Another difference it shows is that solutions of nonlinear equations can “blow-up” even when the equation has no singularities in it. This is in marked contrast with linear equations where the solution will not blow-up if the coefficient and forcing are continuous everywhere.

You might think that the blow-up seen in the last example had something to do with the fact that the stationary point $y = 0$ was bad for our recipe. However, the next example shows that blow-up happens even when there are no stationary points.

Example. Find the solution to the initial-value problem

$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = y_o.$$

Give its interval of existence.

Solution. Because $1 + y^2 > 0$, we see there are no stationary solutions of this equation. Solutions are given implicitly by

$$t = \int \frac{dy}{1 + y^2} = \tan^{-1}(y) + c.$$

The initial condition then implies that

$$0 = \tan^{-1}(y_o) + c,$$

whereby $c = -\tan^{-1}(y_o)$. Here we adopt the common convention that $\frac{\pi}{2} < \tan^{-1}(y_o) < \frac{\pi}{2}$. The solution is therefore given implicitly by

$$t = \tan^{-1}(y) - \tan^{-1}(y_o).$$

This may be solved for y explicitly by first noticing that

$$\tan^{-1}(y) = t + \tan^{-1}(y_o),$$

and then taking the tangent of both sides to obtain

$$y = \tan(t + \tan^{-1}(y_o)).$$

Because \tan becomes singular at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, the interval of existence for this solution is $(\tan^{-1}(y_o) - \frac{\pi}{2}, \tan^{-1}(y_o) + \frac{\pi}{2})$. Notice that this interval contains the initial time $t = 0$ because $\frac{\pi}{2} < \tan^{-1}(y_o) < \frac{\pi}{2}$.

The next example shows another way in which solutions of nonlinear equations can break down.

Example. Find the solution to the initial-value problem

$$\frac{dy}{dt} = -\frac{1}{y}, \quad y(0) = y_o \neq 0.$$

Give its interval of existence.

Solution. Notice that because the right-hand side of the differential equation does not make sense when $y = 0$, no solution may take that value. In particular, we require that $y_o \neq 0$. Because $-1/y \neq 0$, we see there are no stationary solutions of this equation. Solutions are therefore given implicitly by

$$t = - \int y \, dy = -\frac{1}{2}y^2 + c.$$

The initial condition then implies that

$$0 = -\frac{1}{2}y_o^2 + c,$$

whereby $c = \frac{1}{2}y_o^2$. The solution is therefore given implicitly by

$$t = \frac{1}{2}y_o^2 - \frac{1}{2}y^2.$$

This may be solved for y explicitly by first noticing that

$$y^2 = y_o^2 - 2t,$$

and then taking the square root of both sides to obtain

$$y = \pm \sqrt{y_o^2 - 2t}.$$

You must then choose the sign of the square root so that the solution agrees with the initial data — i.e. positive when $y_o > 0$ and negative when $y_o < 0$. In either case you obtain

$$y = \operatorname{sgn}(y_o) \sqrt{y_o^2 - 2t},$$

where

$$\operatorname{sgn}(y_o) = \begin{cases} 1 & \text{if } y_o > 0, \\ 0 & \text{if } y_o = 0, \\ -1 & \text{if } y_o < 0. \end{cases}$$

Finally, notice that to keep the argument of the square root positive we must require that $t < \frac{1}{2}y_o^2$. The interval of existence for this solution is therefore $(-\infty, \frac{1}{2}y_o^2)$.

Remark. In the above example the solution does not blow-up as t approaches $\frac{1}{2}y_o^2$. Indeed, the solution $y(t)$ approaches 0 as t approaches $\frac{1}{2}y_o^2$. However, the derivative of the solution is given by

$$\frac{dy}{dt} = -\frac{1}{y} = -\frac{\operatorname{sgn}(y_o)}{\sqrt{y_o^2 - 2t}},$$

which does blow-up as t approaches $\frac{1}{2}y_o^2$. This happens because the solution approaches the value 0 where the right-hand side of the equation is not defined.

The next example shows another instance in which we must be careful.

Example. Find solutions to the initial-value problem

$$\frac{dy}{dt} = 3y^{\frac{2}{3}}, \quad y(0) = 0.$$

Solution. We see that $y = 0$ is a stationary point of this equation. Therefore $y(t) = 0$ is a stationary solution whose interval of existence is $(-\infty, \infty)$. However, let us carry out our recipe to see where it leads. Nonstationary solutions are given implicitly by

$$t = \int \frac{dy}{3y^{\frac{2}{3}}} = \int \frac{1}{3}y^{-\frac{2}{3}} dy = y^{\frac{1}{3}} + c.$$

Upon solving this for y we find $y = (t - c)^3$ where c is an arbitrary constant. The initial condition then implies that

$$0 = (0 - c)^3 = c^3,$$

whereby $c = 0$. We have thereby found two solutions of the initial-value problem: $y(t) = 0$ and $y(t) = t^3$. As we will now show, there are many others. Let a and b be any two numbers such that

$$a \leq 0 \leq b \quad \text{and} \quad a < b.$$

Then define $y(t)$ by

$$y(t) = \begin{cases} (t - a)^3 & \text{for } t < a, \\ 0 & \text{for } a \leq t < b, \\ (t - b)^3 & \text{for } b \leq t. \end{cases}$$

You will understand this function better if you graph it as we did in class. It is clearly a differentiable function with

$$\frac{dy}{dt}(t) = \begin{cases} 3(t - a)^2 & \text{for } t < a, \\ 0 & \text{for } a \leq t < b, \\ 3(t - b)^2 & \text{for } b \leq t, \end{cases}$$

whereby it clearly satisfies the initial-value problem. Its interval of existence is $(-\infty, \infty)$. Similarly, for every $a \leq 0$ you can construct the solution

$$y(t) = \begin{cases} (t - a)^3 & \text{for } t < a, \\ 0 & \text{for } a \leq t, \end{cases}$$

while for every $b \geq 0$ you can construct the solution

$$y(t) = \begin{cases} 0 & \text{for } t < b, \\ (t - b)^3 & \text{for } b \leq t. \end{cases}$$

Remark. The above example shows a very important difference between nonlinear and linear equations. Specifically, it shows that for nonlinear equations an initial-value problem may not have a unique solution.

3.1.3. Existence and Uniqueness for Autonomous Equations. The non-uniqueness seen in the last example arises because $g(y) = 3y^{\frac{2}{3}}$ does not behave nicely at the stationary point $y = 0$. It is clear that g is continuous at 0, but because $g'(y) = 2y^{-\frac{1}{3}}$ we see that g is not differentiable at 0. The following fact states the differentiability of g is enough to ensure that the solution of the initial-value problem exists and is unique.

Theorem. Let $g(y)$ be a function defined over an open interval (a, b) such that

- g is continuous over (a, b) ,
- g has a finite number of zeros in (a, b) and is differentiable at each of these zeros.

Then for every initial point y_o in (a, b) there exists a unique solution $y = Y(t)$ to the initial-value problem

$$\frac{dy}{dt} = g(y), \quad y(0) = y_o, \quad (3.5)$$

for so long as $Y(t)$ remains within (a, b) . Moreover, this solution is continuously differentiable and is determined by our recipe. Namely, either $g(y_o) = 0$ and $Y(t) = y_o$ is a stationary solution, or $g(y_o) \neq 0$ and $Y(t)$ is a nonstationary solution that satisfies

$$G(Y(t)) = t, \quad Y(0) = y_o,$$

where $G(y)$ is defined by the definite integral

$$G(y) = \int_{y_o}^y \frac{1}{g(z)} dz,$$

whenever the point y is in (a, b) and neither y nor any point between y and y_o is a stationary point of g .

In particular, if g is differentiable over $(-\infty, \infty)$ then the initial-value problem (3.5) has a unique solution that either exists for all time or “blows up” in a finite time. Moreover, this solution is continuously differentiable and is determined by our recipe. This behavior can be seen in the examples above with $g(y) = y^2$ and $g(y) = 1 + y^2$. Indeed, it is seen whenever $g(y)$ is a polynomial.

The above theorem implies that if the initial point y_o lies between two stationary points within (a, b) then the solution $Y(t)$ exists for all time. This is because the uniqueness assertion implies $Y(t)$ cannot cross any stationary point, and is therefore trapped within (a, b) . In particular, if g is differentiable over $(-\infty, \infty)$ then the only solutions that might

“blows up” in a finite time are those that are not trapped above and below by stationary points.

Example. If $g(y) = y^2$ then the only stationary point is $y = 0$. Because $g(y) > 0$ when $y \neq 0$ you see that every nonstationary solution $Y(t)$ will be an increasing function of t . This fact is verified by the formula we derived earlier,

$$Y(t) = \frac{y_o}{1 - y_o t}.$$

When $y_o > 0$ the interval of existence is $(-\infty, 1/y_o)$ and we see that $Y(t) \rightarrow +\infty$ as $t \rightarrow 1/y_o$ while $Y(t) \rightarrow 0$ as $t \rightarrow -\infty$. In this case the solution is trapped below as $t \rightarrow -\infty$ by the stationary point $y = 0$. Similarly, when $y_o < 0$ the interval of existence is $(1/y_o, \infty)$ and we see that $Y(t) \rightarrow -\infty$ as $t \rightarrow 1/y_o$ while $Y(t) \rightarrow 0$ as $t \rightarrow \infty$. In this case the solution is trapped above as $t \rightarrow \infty$ by the stationary point $y = 0$.

You do not need to find an explicit solution to determine the interval of existence of a solution. Rather, you can do it directly from the equation

$$G(y) = t, \quad \text{where} \quad G(y) = \int_{y_o}^y \frac{1}{g(z)} dz.$$

For example, if $g(y) > 0$ over $[y_o, \infty)$ then $G(y)$ will be increasing over $[y_o, \infty)$ and the solution $Y(t)$ will then exist for all $t \geq 0$ such that

$$t < \lim_{y \rightarrow +\infty} G(y).$$

If the above limit is finite then the solution “blows up” as t approaches the value of the right-hand side above. Similarly, if $g(y) > 0$ over $(-\infty, y_o]$ then $G(y)$ will be increasing over $(-\infty, y_o]$ and the solution $Y(t)$ will then exist for all $t \leq 0$ such that

$$\lim_{y \rightarrow -\infty} G(y) < t.$$

If the above limit is finite then the solution “blows down” as t approaches the value of the left-hand side above.

Finally, finding explicit solutions by this our recipe can get complicated even when $g(y)$ is a fairly simple polynomial.

Example. Find the general solution of

$$\frac{dy}{dt} = 4y - y^3.$$

Solution. Because $4y - y^3 = y(2 - y)(2 + y)$, we see that $y = 0$, $y = 2$, and $y = -2$ are stationary solutions of this equation. Nonstationary solutions are given implicitly by

$$t = \int \frac{dy}{4y - y^3}.$$

A partial fraction decomposition yields the identity

$$\frac{1}{4y - y^3} = \frac{1}{y(2 - y)(2 + y)} = \frac{\frac{1}{4}}{y} + \frac{\frac{1}{8}}{2 - y} - \frac{\frac{1}{8}}{2 + y}.$$

(You should be able to write down such partial fraction identities directly.) Hence, solutions are given implicitly by

$$\begin{aligned} t &= \int \frac{\frac{1}{4}}{y} + \frac{\frac{1}{8}}{2 - y} - \frac{\frac{1}{8}}{2 + y} dy = \frac{1}{4} \log(|y|) - \frac{1}{8} \log(|2 - y|) - \frac{1}{8} \log(|2 + y|) + c \\ &= \frac{1}{8} \log(y^2) - \frac{1}{8} \log(|2 - y|) - \frac{1}{8} \log(|2 + y|) + c = \frac{1}{8} \log\left(\frac{y^2}{|4 - y^2|}\right) + c \end{aligned}$$

where c is an arbitrary constant. Upon solving this for y^2 we find

$$8(t - c) = \log\left(\frac{y^2}{|4 - y^2|}\right),$$

which leads to

$$\frac{y^2}{|4 - y^2|} = e^{8(t-c)}.$$

This can then be broken down into two cases.

First, if $y^2 < 4$ then

$$\frac{y^2}{4 - y^2} = e^{8(t-c)},$$

which implies that

$$y^2 = \frac{4}{1 + e^{-8(t-c)}},$$

and finally that

$$y = \pm \sqrt{\frac{4}{1 + e^{-8(t-c)}}}.$$

These solutions exist for every time t . They vanish as $t \rightarrow -\infty$ and approach ± 2 as $t \rightarrow \infty$.

On the other hand, if $y^2 > 4$ then

$$\frac{y^2}{y^2 - 4} = e^{8(t-c)},$$

which implies that

$$y^2 = \frac{4}{1 - e^{-8(t-c)}}.$$

So long as the denominator is positive, we find the solutions

$$y = \pm \sqrt{\frac{4}{1 - e^{-8(t-c)}}}.$$

The denominator is positive if and only if $t > c$. Therefore these solutions exist for every time $t > c$. They diverge (blow-up) to $\pm\infty$ as $t \rightarrow c^+$ and approach ± 2 as $t \rightarrow \infty$.

We therefore have found the stationary solutions $y = 0$, $y = 2$, and $y = -2$, and the nonstationary solutions

$$\begin{aligned} y &= -\sqrt{\frac{4}{1 - e^{-8(t-c)}}} && \text{when } -\infty < y < -2 \text{ and } t > c, \\ y &= -\sqrt{\frac{4}{1 + e^{-8(t-c)}}} && \text{when } -2 < y < 0, \\ y &= +\sqrt{\frac{4}{1 + e^{-8(t-c)}}} && \text{when } 0 < y < 2, \\ y &= +\sqrt{\frac{4}{1 - e^{-8(t-c)}}} && \text{when } 2 < y < \infty \text{ and } t > c, \end{aligned}$$

where c is an arbitrary constant. This list includes every solution of the equation. \square