First-Order Ordinary Differntial Equations: Classification and Linear Equations

David Levermore
Department of Mathematics
University of Maryland

1 February 2009

These notes cover some of the material that we covered in class on first-order ordinary differential equations. As the presentation of this material in class was somewhat different from that in the book, I felt that a written review closely following the class presentation might be appreciated.

1. Introduction, Classification, and Overview

1.1. Introduction. A *differential equation* is an algebraic relation involving derivatives of one or more unknown functions with respect to one or more independent variables, and possibly either the unknown functions themselves or their independent variables, that hold at each point in the domain of those functions.

For example, an unknown function $p(t)$ might satisfy the relation

$$\frac{dp}{dt} = 5p\,. \tag{1.1}$$

This is a differential equation because it involves the derivative of the unknown function p. It also involves the value of p, but not the independent variable t. It is understood that this relation should hold every point t where $p(t)$ and its derivative are defined.

Similarly, unknown functions $u(x,y)$ and $v(x,y)$ might satisfy the relation

$$\partial_x u + \partial_y v = 0\,, \tag{1.2}$$

where $\partial_x u$ and $\partial_y v$ denote partial derivatives. This is a differential equation because it involves derivatives of the unknown functions u and v. It does not involve either the values of u and v or the independent variables x and y. It is understood that this relation should hold every point (x,y) where $u(x,y)$, $v(x,y)$ and their partial derivatives appearing in (1.2) are defined.

Here are other examples of differential equations that involve derivatives of a single unkown function:

$$\begin{array}{llll}
\text{(a)} & \dfrac{dv}{dt} = 9.8 - .1v^2\,, & \text{(b)} & \dfrac{d^2\theta}{dt^2} + \sin(\theta) = 0\,, \\[2ex]
\text{(c)} & \left(\dfrac{dy}{dx}\right)^2 + x^2 + y^2 = -1\,, & \text{(d)} & \left(\dfrac{dy}{dx}\right)^2 + 4y^4 = 1\,, \\[2ex]
\text{(e)} & \dfrac{d^2u}{dr^2} + \dfrac{2}{r}\dfrac{du}{dr} + u = 0\,, & \text{(f)} & \dfrac{d^3u}{dx^3} + u\dfrac{du}{dx} + \dfrac{du}{dx} = 0\,, \\[2ex]
\text{(g)} & \dfrac{d^2x}{dt^2} + 9x = \cos(2t)\,, & \text{(h)} & \dfrac{d^2r}{dt^2} = \dfrac{1}{r^4} - \dfrac{1}{r^2}\,, \\[2ex]
\text{(i)} & \dfrac{d^2A}{dx} + xA = 0\,, & \text{(j)} & W^3\dfrac{dW}{dz} + \frac{1}{6}z = 0\,, \\[2ex]
\text{(k)} & \partial_{tt}h - \partial_{xx}h - \partial_{yy}h = 0\,, & \text{(l)} & \partial_{xx}\phi + \partial_{yy}\phi + \partial_{zz}\phi = 0\,, \\[2ex]
\text{(m)} & \partial_t u + u\partial_x u = \partial_{xx}u\,, & \text{(n)} & \partial_t T = \partial_x\left[T^4\partial_x T\right]\,.
\end{array} \tag{1.3}$$

In all of these examples except k and l the unknown function itself also appears in the equation. In examples d, e, g, i, and j the independent variable also appears in the equation.

1.2. Classification. A differential equation is called an *ordinary differential equation* (ODE) if it invloves derivatives with respect to only one independent variable. Otherwise, it is called a *partial differential equation* (PDE). Example (1.1) is an ordinary differential equation. Example (1.2) is a partial differential equation. Of the examples in (1.3):

$$\begin{aligned} \text{a} - \text{j} \quad & \text{are ordinary differential equations}\,; \\ \text{k} - \text{n} \quad & \text{are partial differential equations}\,. \end{aligned}$$

The *order* of a differential equation is the order of the highest derivative that appears in it. An nth-order differential equation is one whose order is n. Examples (1.1) and (1.2) are both first-order differential equations. Of the examples in (1.3):

$$\begin{aligned} \text{a, c, d, j} \quad & \text{are first-order differential equations}\,; \\ \text{b, e, g, h. i, k, l, m, n} \quad & \text{are second-order differential equations}\,; \\ \text{f} \quad & \text{is a third-order differential equation}\,. \end{aligned}$$

A differential equation is said to be *linear* if each side of the equation is a sum of terms, each of which either

- is a derivative of an unknown function times a factor that is independent of the unknown functions,
- is an unknown function times a factor that is independent of the unknown fuctions,
- or is entirely independent of the unknown fuctions.

Otherwise it is said to be *nonlinear*. Examples (1.1) and (1.2) are both linear differential equations. Of the examples in (1.3):

$$\begin{aligned} \text{e, g, i, k, l} \quad & \text{are linear differential equations}\,; \\ \text{a} - \text{d, f, h, j, m, n} \quad & \text{are nonlinear differential equations}\,. \end{aligned}$$

Every n^{th} order linear ordinary differential equation for a single unknown function $y(t)$ can be brought into the form

$$p_0(t)\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = r(t)\,,$$

where $p_0(t)$, $p_2(t)$, $\cdots$, $p_n(t)$, and $r(t)$ are given functions of t such that $p_0(t) \neq 0$. Linear differential equations are important because much more can be said about them than for general nonlinear differential equations.

In applications one is often faced with a *system* of coupled differential equations — typically a system of m equations for m unknown functions. For example, two unknown functions $p(t)$ and $q(t)$ might satisfy the system

$$\frac{dp}{dt} = (6 - 2p - q)p\,, \qquad \frac{dq}{dt} = (8 - 4p - q)p\,. \tag{1.4}$$

Similarly, two unknown functions $u(x, y)$ and $v(x, y)$ might satisfy the system

$$\partial_x u = \partial_y v\,, \qquad \partial_y u + \partial_x v = 0\,. \tag{1.5}$$

The order of a system of differential equations is the order of the highest derivatrive appearing in the entire system. Example (1.4) is a first-order system of ordinary differential equations, while (1.5) is a first-order system of partial differential equations. The size of the systems that arise in applications can be extremely large. Systems of 10^8 ordinary differential equations are being solved every day.

1.3. Course Overview. Differential equations arise in mathematics, physics, chemistry, biology, medicine, pharmacology, communications, electronics, finance, economics, areospace, meteorology, climatology, oil recovery, hydrology, ecology, combustion, image processing, and in many other fields. Partial differential equations are at the heart of most of these applications. You need to know something about ordinary differential equations before you study partial differential equations. This course will serve as your introduction to ordinary differential equations. More specifically, we will study four classes of ordinary differential equations. We illustrate these four classes below denoting the independent variable by t.

(I) We will begin with single first-order ODEs that can be brought into the form

$$\frac{dy}{dt} = f(t, y)\,. \tag{1.6}$$

These will be covered before the first in-class exam. You may have seen some of the material in your calculus courses.

(II) We will next study single n^{th}-order linear ODEs that can be brought into the form

$$\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = f(t)\,. \tag{1.7}$$

These will be covered before the second in-class exam. This is the heart of the course. Many students find this the most difficult part of the course.

(III) We will then turn towards systems of n first-order linear ODEs that can brought into the form

$$\begin{aligned}
\frac{dy_1}{dt} &= a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + f_1(t)\,,\\
\frac{dy_2}{dt} &= a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + f_2(t)\,,\\
&\vdots\\
\frac{dy_n}{dt} &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + f_n(t)\,;
\end{aligned} \tag{1.8}$$

These will be covered before the third in-class exam. This material builds upon the material covered in part II.

(IV) Finally, we will study systems of two first-order ODEs that can brought into the form

$$\frac{dx}{dt} = f(x, y)\,, \qquad \frac{dy}{dt} = g(x, y)\,. \tag{1.9}$$

These will be covered before and immediately after the third in-class exam. This material builds upon the material in parts I and III.

This is far from a complete treatment of the subject. It will however prepare you to learn more about ordinary differential equations or to learn about partial differential equations.

2. First-Order Equations: Explict and Linear

2.1. Introduction. We now begin our study of first-order ordinary differential equations that involve a single real-valued unknown function $y(t)$. These can always be brought into the form

$$F\left(t, y, \frac{dy}{dt}\right) = 0\,.$$

If we try to solve this equation for dy/dt in terms of t and y then there might be no solutions or many solutions. For example, equation (c) of (1.3) clearly has no (real) solutions because the sum of nonnegative terms cannot add to -1. On the other hand, equation (d) will be satisfied if either

$$\frac{dy}{dx} = \sqrt{1-4y^2}\,, \qquad \text{or} \qquad \frac{dy}{dx} = -\sqrt{1-4y^2}\,.$$

To avoid these complications we will restrict ourselves to equations that can be brought into the form

$$\frac{dy}{dt} = f(t, y)\,. \tag{2.1}$$

Examples (1.1) and (a) of (1.3) are already in this form. Example (j) of (1.3) can easily be brought into this form. And as we saw above, example (d) of (1.3) can be reduced to two equations in this form.

We will say that $y = Y(t)$ is a *solution* of (2.1) over an interval (a, b) whenever

$$\begin{aligned} &\text{(i)} && \text{the function } Y \text{ is differentiable over } (a,b)\,,\\ &\text{(ii)} && f(t, Y(t)) \text{ is defined for every } t \text{ in } (a,b)\,,\\ &\text{(iii)} && Y'(t) = f(t, Y(t)) \text{ for every } t \text{ in } (a,b)\,. \end{aligned} \tag{2.2}$$

Some basic questions we want to address are the following.

- When does (2.1) have solutions?
- Under what conditions is a solution unique?
- How can we find analytic expressions for solutions?

- How can we visualize solutions?
- How can we approximate solutions?

We will focus on the last three questions. They address practical skills that you can apply when faced with a differential equation. The first two questions will be viewed through the lens of the last three. They are important because differential equations that arise in applications are supposed to model or predict something. If an equation either does not have solutions or has more than one solution then it fails to meet this objective. Moreover, in those situations the methods by which we will address the last three questions can give misleading results. We will therefore study the first two questions with an eye towards avoiding such pitfalls. Rather than addressing these questions for a general $f(t, y)$ in (2.1), we will start by treating special forms $f(t, y)$ of increasing complexity.

2.2. Explicit Equations. The simplest form of (2.1) to treat is that of so-called *explicit* equations,

$$\frac{dy}{dt} = f(t)\,. \tag{2.3}$$

In this case the deriviative is given as an explicit function of t. This case is usually covered in calculus courses, so we only review it here.

2.2.1. Recipe for Solving Explicit Equations. You should recall that a differentiable function F is said to be a *primitive* or *antiderivative* of f if $F' = f$. You thereby see that $y = Y(t)$ will be a solution of (2.3) if and only if Y is a primitive of f. You should also recall that if you know one primitive F of f then any other primitive Y of f must have the form $Y(t) = F(t) + c$ for some constant c. We thereby see that if (2.3) has one solution then it has a family of solutions given by the indefinite integral of f — namely, by

$$y = \int f(t)\,dt = F(t) + c\,, \quad \text{where } F'(t) = f(t) \text{ and } c \text{ is any constant}\,. \tag{2.4}$$

Moreover, there are no other solutions of (2.3). The family (2.4) is therefore called a *general solution* of the differential equation (2.3).

2.2.2. Initial-Value Problems for Explicit Equations. In order to pick a unique solution from the family (2.3) one must impose an additional condition that determines c. We do this by imposing a so-called *initial condition* of the form

$$y(t_I) = y_I\,,$$

where t_I is called the *initial time* and y_I is called the *initial value* or *initial datum*. The combination of the differential equation (2.2) with the above initial condition is

$$\frac{dy}{dt} = f(t)\,, \qquad y(t_I) = y_I\,. \tag{2.5}$$

This is a so-called an *initial-value problem*. By imposing the initial condition upon the family (2.4) we see that

$$F(t_I) + c = y_I\,,$$

which implies that $c = y_I - F(t_I)$. Therefore, if f has a primitive F then the unique solution of initial-value problem (2.5) is given by

$$y = y_I + F(t) - F(t_I)\,. \tag{2.6}$$

The above arguments show that the problems of finding either a general solution of (2.3) or the unique solution of the initial-value problem (2.5) reduce to the problem of finding a primitive F of f. Given such an F, a general solution of (2.3) is given by (2.4) while the unique solition of initial-value problem (2.5) is given by (2.6). These arguments however do not insure that such a primitive exists. Of course, for sufficiently simple f you can find a primitive analytically.

Example: Find a general solution to the differential equation

$$\frac{dw}{dx} = 6x^2 + 1\,.$$

Solution: By (2.4) a general solution is

$$w = \int \left(6x^2 + 1\right) dx = 2x^3 + x + c\,.$$

Example: Find a solution to the initial-value problem

$$\frac{dw}{dx} = 6x^2 + 1\,, \qquad w(1) = 5\,.$$

Solution: The previous example shows the solution has the form $w = 2x^3 + x + c$ for some constant c. Imposing the initial condition gives $2 \cdot 1^3 + 1 + c = 5$, which implies $c = 2$. Hence, the solution is $w = 2x^3 + x + 2$.

Alternative Solution: By (2.6) with $x_I = 1$, $w_I = 5$, and the primitive $F(x) = 2x^3 + x$ we find

$$\begin{aligned} w &= w_I + F(x) - F(x_I) = 5 + F(x) - F(1) \\ &= 5 + \left(2x^3 + x\right) - \left(2 \cdot 1^3 + 1\right) = 2x^3 + x + 2\,. \end{aligned}$$

Remark. As the solutions to previous example illustrate, when solving an initial-value problem it is often easier to first find a general solution and then evaluate the c from the initial condition rather than to directly apply formula (2.6). With that approach you do not have to memorize formula (2.6).

2.2.3. Existence of Solutions for Explicit Equations. Finally, even when you cannot find a primitive analytically, you can show that a solution exists by appealing to the Second

Fundamental Theorem of Calculus. It states that if f is continuous over an interval (a, b) then for every t_o in (a, b) one has

$$\frac{d}{dt}\int_{t_o}^{t} f(s)\,ds = f(t)\,.$$

In other words, f has a primitive over (a, b) that can be expressed as a definite integral. Here s is the "dummy" variable of integration in the above definite integral. If t_I is in (a, b) then the First Fundamental Theorem of Calculus implies that formula (2.6) can be expressed as

$$y = y_I + \int_{t_I}^{t} f(s)\,ds\,. \tag{2.7}$$

This shows that if f is continuous over an interval (a, b) that contains t_I then the initial-value problem (2.5) has a unique solution over (a, b), which is given by formula (2.7). This formula can be approximated by numerical quadrature for any such f.

> **Definition:** The largest interval over which a solution exists is called its *interval of existence* or *interval of definition.*

For explicit equations one can usually identify the interval of existence for the solution of the initial-value problem (2.5) by simply looking at $f(t)$. Specifically, if $Y(t)$ is the solution of the initial value problem (2.5) then its interval of existence will be (t_L, t_R) whenever:

- $f(t)$ is continuous over (t_L, t_R),
- the initial time t_I is in (t_L, t_R),
- $f(t)$ is not defined at both $t = t_L$ and $t = t_R$.

This is because the first two bullets along with the formula (2.7) imply that the interval of existence will be at least (t_L, t_R), while the last two bullets along with our definition (2.2) of solution imply that the interval of existence can be no bigger than (t_L, t_R). This argument works when $t_L = -\infty$ or $t_R = \infty$.

2.3. Linear Equations. The next simplest form of (2.1) to treat is that of so-called linear equations,

$$\frac{dy}{dt} = f(t) - a(t)y\,.$$

In this case the derivative of y is given as a linear function of y whose coefficients are functions of t. It contains the explicit case when $a(t) = 0$.

More generally, every linear first-order ODE for a single unknown function $y(t)$ can be brought into the form

$$p(t)\frac{dy}{dt} + q(t)y = r(t)\,, \tag{2.8}$$

where $p(t)$, $q(t)$, and $r(t)$ are given functions of t such that $p(t) \neq 0$ for those t over which the equation is considered. The functions $p(t)$ and $q(t)$ are called *coefficients* while the function $r(t)$ is called the *forcing* or *driving.*

A linear equation may not be given to you in the form (2.8). However, you can put it into this form by simply grouping all the terms involving either the unkown function or its derivative on the left-hand side, while grouping all the other terms on the right-hand side.

Example. Consider the equation

$$e^t \frac{dz}{dt} + t^2 = \frac{2t+z}{1+t^2}\,.$$

You should be able to see that this equation is linear and to bring it into the form (2.8).

Solution. By grouping all the terms involving either the z or its derivative on the left-hand side, while grouping all the other terms on the right-hand side, we obtain

$$e^t \frac{dz}{dt} + \frac{1}{1+t^2}\, z = \frac{2t}{1+t^2} - t^2\,.$$

This is in the form (2.8). $\square$

2.3.1. Recipe for Solving Linear Equations. The following is a straightforward recipe that reduces the problem of generating an analytic solution of (2.8) to that of finding two primitives. One first brings (2.8) into the *normal form* by dividing by $p(t)$. This yields

$$\frac{dy}{dt} + a(t)y = f(t)\,, \qquad \text{where} \quad a(t) = \frac{q(t)}{p(t)}\,, \quad f(t) = \frac{r(t)}{p(t)}\,. \tag{2.9}$$

Below we will show that this is equivalent to the so-called *integrating factor form*

$$\frac{d}{dt}\Big(e^{A(t)}y\Big) = e^{A(t)} f(t)\,, \qquad \text{where } A'(t) = a(t)\,. \tag{2.10}$$

This is an explicit equation for the derivative of $e^{A(t)}y$ that can be integrated to obtain

$$e^{A(t)}y = \int e^{A(t)} f(t)\, dt = B(t) + c\,, \qquad \text{where } B'(t) = e^{A(t)} f(t) \text{ and } c \text{ is any constant}\,. \tag{2.11}$$

A general solution of (2.8) is therefore given by the family

$$y = e^{-A(t)} B(t) + e^{-A(t)} c\,. \tag{2.12}$$

If you are solving an initial-value problem then you can evaluate c from the initial condition.

The key to understanding the above recipe is to understand the equivalence of the normal form (2.9) and integrating factor form (2.10). This equivalence follows from the fact that

$$\begin{aligned}\frac{d}{dt}\Big(e^{A(t)}y\Big) &= e^{A(t)}\frac{dy}{dt} + \frac{d}{dt}\Big(e^{A(t)}\Big)y \\ &= e^{A(t)}\frac{dy}{dt} + e^{A(t)}A'(t)y = e^{A(t)}\left(\frac{dy}{dt} + a(t)y\right).\end{aligned}$$

This calculation shows that equation (2.10) is simply equation (2.9) multiplied by $e^{A(t)}$. Because the factor $e^{A(t)}$ is always positive, the equations are therefore equivalent. We call $e^{A(t)}$ an *integrating factor* of equation (2.9) because after multiplying both sides of (2.9) by $e^{A(t)}$ the left-hand side can be written as the derivative of $e^{A(t)}y$. An integrating factor thereby allows you to reduce the linear case to the explicit case.

Rather than memorizing formula (2.12), it is easier to approach first-order linear ordinary differential equations by simply retracing the steps by which (2.12) was derived. We illustrate this approach with the following examples.

Example. Find the general solution to

$$\frac{dx}{dt} = -5x + e^{2t}\,.$$

Solution. First bring the equation into the normal form

$$\frac{dx}{dt} + 5x = e^{2t}\,.$$

An integrating factor is $e^{A(t)}$ where $A'(t) = 5$. Setting $A(t) = 5t$, we then bring the equation into the integrating factor form

$$\frac{d}{dt}\Big(e^{5t}x\Big) = e^{5t}e^{2t} = e^{7t}\,.$$

By integrating both sides of this equation we obtain

$$e^{5t}x = \int e^{7t}\,dt = \tfrac{1}{7}e^{7t} + c\,.$$

The general solution is therefore given by

$$x = \tfrac{1}{7}e^{2t} + e^{-5t}c\,.$$

Example. Find the general solution to

$$(1+t^2)\frac{dz}{dt} + 4tz = \frac{1}{\big(1+t^2\big)^2}\,.$$

Solution. First bring the equation into the normal form

$$\frac{dz}{dt} + \frac{4t}{1+t^2}z = \frac{1}{\big(1+t^2\big)^3}\,.$$

An integrating factor is $e^{A(t)}$ where $A'(t) = 4t/(1+t^2)$. Setting $A(t) = 2\log(1+t^2)$, we see that

$$e^{A(t)} = e^{2\log(1+t^2)} = \left(e^{\log(1+t^2)}\right)^2 = \left(1+t^2\right)^2 .$$

We then bring the differential equation into the integrating factor form

$$\frac{d}{dt}\Big((1+t^2)^2 z\Big) = \left(1+t^2\right)^2 \frac{1}{\left(1+t^2\right)^3} = \frac{1}{1+t^2} .$$

By integrating both sides of this equation we obtain

$$(1+t^2)^2 z = \int \frac{1}{1+t^2}\, dt = \tan^{-1}(t) + c .$$

The general solution is therefore given by

$$z = \frac{\tan^{-1}(t)}{\left(1+t^2\right)^2} + \frac{c}{\left(1+t^2\right)^2} .$$

2.3.2. Homogeneous Linear Equations. The linear equation (2.8) is said to be *homogeneous* when $r(t) = 0$ for every t, and is said to be *inhomogeneous* otherwise. When (2.8) is homogeneous its normal form (2.9) is simply

$$\frac{dy}{dt} + a(t)y = 0 , \qquad \text{where} \quad a(t) = \frac{q(t)}{p(t)} . \tag{2.13}$$

By formula (2.12) its general solution is simply

$$y = e^{-A(t)} c , \qquad \text{where } A'(t) = a(t) \text{ and } c \text{ is any constant} . \tag{2.14}$$

Hence, for homogenous linear equations the recipe for solution only requires finding one primitive — namely, a primitive of $a(t)$. This means that for simply enough $a(t)$ you should be able to write down general solutions immediately.

Example. Find the general solution to

$$\frac{dp}{dt} = 5p .$$

Solution. Because $a(t) = -5$, a general solution is given by

$$p = e^{5t} c , \qquad \text{where } c \text{ is an arbitrary constant} .$$

Example. Find the general solution to

$$\frac{dz}{dt} + t^2 z = 0\,.$$

Solution. Because $a(t) = t^2$, a general solution is given by

$$z = e^{-\frac{1}{3}t^3} c\,, \qquad \text{where } c \text{ is an arbitrary constant}\,.$$

2.3.3. Initial-Value Problems for Linear Equations. In order to pick a unique solution from the family (2.12) one must impose an additional condition that determines c. We do this by again imposing an *initial condition* of the form

$$y(t_I) = y_I\,,$$

where t_I is called the *initial time* and y_I is called the *initial value* or *initial datum*. The combination of the differential equation (2.9) with the above initial condition is

$$\frac{dy}{dt} + a(t)y = f(t)\,, \qquad y(t_I) = y_I\,. \tag{2.15}$$

This is a so-called an *initial-value problem*. By imposing the initial condition upon the family (2.12) we see that

$$e^{-A(t_I)}B(t_I) + e^{-A(t_I)}c = y_I\,,$$

which implies that $c = e^{A(t_I)}y_I - B(t_I)$. Therefore, if the primitives $A(t)$ and $B(t)$ exist then the unique solution of initial-value problem (2.15) is given by

$$y = e^{-A(t)+A(t_I)}y_I + e^{-A(t)}\big(B(t) - B(t_I)\big)\,. \tag{2.16}$$

2.3.4. Existence of Solutions for Linear Equations. Even when you cannot find primitives $A(t)$ and $B(t)$ analytically, you can show that a solution exists by appealing to the Second Fundamental Theorem of Calculus whenever $a(t)$ and $f(t)$ are continuous over an interval (t_L, t_R) that contains the initial time t_I. In that case one can express $A(t)$ and $B(t)$ as the definite integrals

$$A(t) = \int_{t_I}^{t} a(s)\,ds\,, \qquad B(t) = \int_{t_I}^{t} e^{A(s)} f(s)\,ds\,.$$

For this choice of $A(t)$ and $B(t)$ one has $A(t_I) = B(t_I) = 0$, whereby formula (2.16) becomes

$$y = e^{-A(t)}y_I + e^{-A(t)}B(t) = e^{-A(t)}y_I + \int_{t_I}^{t} e^{-A(t)+A(s)} f(s)\,ds\,. \tag{2.17}$$

The First Fundamental Theorem of Calculus implies that

$$A(t) - A(s) = \int_s^t a(s_1)\, ds_1 \,,$$

whereby formula (2.17) can be expressed as

$$y = \exp\Big(- \int_{t_I}^t a(s)\, ds \Big) y_I + \int_{t_I}^t \exp\Big(- \int_s^t a(s_1)\, ds_1 \Big) f(s)\, ds \,. \tag{2.18}$$

This shows that if a and f are continuous over an interval (t_L, t_R) that contains t_I then the initial-value problem (2.15) has a unique solution over (t_L, t_R), which is given by formula (2.18).

For linear equations one can usually identify the interval of existence for the solution of the initial-value problem (2.15) by simply looking at $a(t)$ and $f(t)$. Specifically, if $Y(t)$ is the solution of the initial value problem (2.15) then its interval of existence will be (t_L, t_R) whenever:

- the coefficient $a(t)$ and forcing $f(t)$ are continuous over (t_L, t_R),
- the initial time t_I is in (t_L, t_R),
- either the coefficient $a(t)$ or the forcing $f(t)$ is not defined at both $t = t_L$ and $t = t_R$.

This is because the first two bullets along with the formula (2.18) imply that the interval of existence will be at least (t_L, t_R), while the last two bullets along with our definition (2.2) of solution imply that the interval of existence can be no bigger than (t_L, t_R) because the equation breaks down at $t = t_L$ and $t = t_R$. This argument works when $t_L = -\infty$ or $t_R = \infty$.

Example: Give the interval of existence for the solution of the initial-value problem

$$\frac{dz}{dt} + \cot(t)\, z = \frac{1}{\log(t^2)} \,, \qquad z(4) = 3 \,.$$

Solution: The coefficient $\cot(t)$ is not defined at $t = n\pi$ where n is any integer, and is continuous everywhere else. The forcing $1/\log(t^2)$ is not defined at $t = 0$ and $t = 1$, and is continuous everywhere else. The interval of existence is therefore $(\pi, 2\pi)$ because: both $\cot(t)$ and $1/\log(t^2)$ are continuous over this interval; the initial time is $t = 4$, which is in this interval; $\cot(t)$ is not defined at $t = \pi$ and $t = 2\pi$. $\square$

Example: Give the interval of existence for the solution of the initial-value problem

$$\frac{dz}{dt} + \cot(t)\, z = \frac{1}{\log(t^2)} \,, \qquad z(2) = 3 \,.$$

Solution: The interval of existence is $(1, \pi)$ because: both $\cot(t)$ and $1/\log(t^2)$ are continuous over this interval; the initial time is $t = 2$, which is in this interval; $\cot(t)$ is not defined at $t = \pi$ while $1/\log(t^2)$ is not defined at $t = 1$. $\square$

Remark: If $y = Y(t)$ is a solution of (2.15) whose interval of existence is (t_L, t_R) then this does not mean that $Y(t)$ will become singular at either $t = t_L$ or $t = t_R$ when those endpoints are finite. For example, $y = t^4$ solves the initial-value problem

$$t\,\frac{dy}{dt} - 4y = 0\,, \qquad y(1) = 1\,,$$

and is defined for every t. However, the interval of existence is just $(0, \infty)$ because the initial time is $t = 1$ and normal form of the equation is

$$\frac{dy}{dt} - \frac{4}{t}\,y = 0\,,$$

the coefficient of which is undefined at $t = 0$.

Remark: It is natural to ask why we do not extend our definition of solutions so that $y = t^4$ is considered a solution of the initial-value problem

$$t\,\frac{dy}{dt} - 4y = 0\,, \qquad y(1) = 1\,,$$

for every t. For example, we might say that $y = Y(t)$ is a solution provided it is differentiable and satisfies the above equation rather than its normal form. However by this definition the function

$$Y(t) = \begin{cases} t^4 & \text{for } t \geq 0 \\ ct^4 & \text{for } t < 0 \end{cases}$$

also solves the initial-value problem for any c. This shows that because the equation breaks down at $t = 0$, there are many ways to extend the solution $y = t^4$ to $t < 0$. We avoid such complications by requiring the normal form of the equation to be defined.