HIGHER-ORDER LINEAR
ORDINARY DIFFERENTIAL EQUATIONS II:
Constant Coefficient Nonhomogeneous and Vibrations

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Because the presentation of this material in class will differ from that in the book, I felt that notes that closely follow the class presentation might be appreciated.

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4. Nonhomogeneous Equations

4.1: General Theory. We are now ready to study nonhomogeneous linear equations. An n^{th} order nonhomogeneous linear ODE has the normal form

$$\mathrm{L}(t)y = f(t)\,, \tag{4.1}$$

where the differential operator $\mathrm{L}(t)$ has the form

$$\mathrm{L}(t) = \frac{d^n}{dt^n} + a_1(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{d}{dt} + a_n(t)\,. \tag{4.2}$$

We will assume throughout this section that the coefficients $a_1, a_2, \cdots, a_n$ and the forcing f are continuous over an interval (t_L, t_R), so that Therorem 1.1 can be applied.

We will exploit the following properties of nonhomogeneous equations.

Theorem 4.1: If $Y_1(t)$ and $Y_2(t)$ are solutions of (4.1) then $Z(t) = Y_1(t) - Y_2(t)$ is a solution of the associated homogeneous equation $\mathrm{L}(t)Z(t) = 0$.

Proof: Because $\mathrm{L}(t)Y_1(t) = f(t)$ and $\mathrm{L}(t)Y_2(t) = f(t)$ one sees that

$$\mathrm{L}(t)Z(t) = \mathrm{L}(t)\big(Y_1(t) - Y_2(t)\big) = \mathrm{L}(t)Y_1(t) - \mathrm{L}(t)Y_2(t) = f(t) - f(t) = 0\,.$$

Theorem 4.2: If $Y_P(t)$ is a solution of (4.1) and $Y_H(t)$ is a solution of the associated homogeneous equation $\mathrm{L}(t)Y_H(t) = 0$ then $Y(t) = Y_H(t) + Y_P(t)$ is also a solution of (4.1).

Proof: Because $\mathrm{L}(t)Y_H(t) = 0$ and $\mathrm{L}(t)Y_P(t) = f(t)$ one sees that

$$\mathrm{L}(t)Y(t) = \mathrm{L}(t)\big(Y_H(t) + Y_P(t)\big) = \mathrm{L}(t)Y_H(t) + \mathrm{L}(t)Y_P(t) = 0 + f(t) = f(t)\,.$$

Theorem 4.2 suggests that we can construct general solutions of the nonhomogeneous equation (4.1) as follows.

(1) Find a general solution $Y_H(t)$ of the *associated homogeneous equation* $\mathrm{L}(t)y = 0$.

(2) Find a *particular solution* $Y_P(t)$ of equation (4.1).

(3) Then $Y_H(t) + Y_P(t)$ is a general solution of (4.1).

Of course, step (1) reduces to finding a fundamental set of solutions of the associated homogeneous equation, $Y_1, Y_2, \cdots, Y_n$. Then

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t) + \cdots + c_nY_n(t)\,.$$

If $\mathrm{L}(t)$ has constant coefficients (so that $\mathrm{L}(t) = \mathrm{L}$) then this can be done by the recipe of Section 3.

Example. One can check that $\frac{1}{4}t$ is a particular solution of

$$\mathrm{D}^2 y + 4y = t\,.$$

This equation has constant coefficients. Its characteristic polynomial is $p(z) = z^2 + 4$, which has roots $\pm i2$. A general solution is therefore

$$y = c_1 \cos(2t) + c_2 \sin(2t) + \tfrac{1}{4}t\,.$$

Example. One can check that $-\frac{1}{2}e^t$ is a particular solution of

$$\mathrm{D}^2 y - \mathrm{D}y - 2y = e^t\,.$$

This equation has constant coefficients. Its characteristic polynomial is $p(z) = z^2 - z - 2 = (z-2)(z+1)$, which has roots -1 and 2. A general solution is therefore

$$y = c_1 e^{-t} + c_2 c^{2t} - \tfrac{1}{2}e^t\,.$$

These examples show that when $\mathrm{L}(t)$ has constant coefficients (so that $\mathrm{L}(t) = \mathrm{L}$), finding $Y_P(t)$ becomes the crux of matter. If $\mathrm{L}(t)$ does not have constant coefficients then a fundamental set of solutions of the associated homogeneous equation will generally be given to you. In that case, finding $Y_P(t)$ again becomes the crux of matter. The remainder of this section will present methods for finding particular solutions $Y_P(t)$.

4.2: Undertermined Coefficients. This method can be used to construct a particular solution of an n^{th} order nonhomogeneous linear ODE in the normal form

$$\mathrm{L}y = f(t) \tag{4.3}$$

whenever the following two conditions are met.

(1) The differential operator L has constant coefficients,

$$\mathrm{L} = \mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n\,. \tag{4.4}$$

(2) The forcing $f(t)$ has the form

$$\begin{aligned} f(t) = \big(f_0 t^d + f_1 t^{d-1} + \cdots + f_d\big)e^{rt}\cos(st) \\ + \big(g_0 t^d + g_1 t^{d-1} + \cdots + g_d\big)e^{rt}\sin(st)\,, \end{aligned} \tag{4.5}$$

for some nonnegative integer d and real numbers r and s. Here we are assuming that $f_0,\ f_1,\ \cdots,\ f_d$ and $g_0,\ g_1,\ \cdots,\ g_d$ are all real and that either f_0 or sg_0 is nonzero. The integer d is called the *degree* of the forcing while the number $r + is$ is called its *characteristic.*

The first of these conditions is always easy to verify by inspection. Verification of the second usually can also be done by inspection, but sometimes it might require the use of a trigonometric or some other identity. You should be able identify when a forcing $f(t)$ has the form (4.5), and when it is, to read-off its degree and characteristic.

Example: The forcing of the equation $\mathrm{L}y = 2e^{2t}$ has the form (4.5) with degree $d = 0$ and characteristic $r + is = 2$.

Example: The forcing of the equation $\mathrm{L}y = t^2 e^{-3t}$ has the form (4.5) with degree $d = 2$ and characteristic $r + is = -3$.

Example: The forcing of the equation $\mathrm{L}y = t\, e^{5t} \sin(3t)$ has the form (4.5) with degree $d = 1$ and characteristic $r + is = 5 + i3$.

Example: The forcing of the equation $\mathrm{L}y = \sin(2t)\cos(2t)$ can be put into the form (4.5) by using the double-angle identity $\sin(4t) = 2\sin(2t)\cos(2t)$. The equation can thereby be expressed as $\mathrm{L}y = \frac{1}{2}\sin(4t)$. The forcing now has the form (4.5) with degree $d = 0$ and characteristic $r + is = i4$.

The method of undetermined coefficients is based on the observation that for any forcing of the form (4.5) one can construct explicit formulas for a particular solution of (4.3) by evaluating the KEY identity and some of its derivatives with respect to z at $z = r + is$. For example, if $p(z)$ is the characteristic polynomial of L then the KEY identity and its first four derivatives are

$$
\begin{aligned}
\mathrm{L}\big(e^{zt}\big) &= p(z) e^{zt} \,, \\
\mathrm{L}\big(t\, e^{zt}\big) &= p(z)\, t\, e^{zt} + p'(z)\, e^{zt} \,, \\
\mathrm{L}\big(t^2 e^{zt}\big) &= p(z)\, t^2 e^{zt} + 2p'(z)\, t\, e^{zt} + p''(z)\, e^{zt} \,, \\
\mathrm{L}\big(t^3 e^{zt}\big) &= p(z)\, t^3 e^{zt} + 3p'(z)\, t^2 e^{zt} + 3p''(z)\, t\, e^{zt} + p'''(z)\, e^{zt} \,, \\
\mathrm{L}\big(t^4 e^{zt}\big) &= p(z)\, t^4 e^{zt} + 4p'(z)\, t^3 e^{zt} + 6p''(z)\, t^2 e^{zt} + 4p'''(z)\, t\, e^{zt} + p^{(4)}(z)\, e^{zt} \,.
\end{aligned}
\tag{4.6}
$$

Notice that when these are evaluated at $z = r + is$ then the terms on the right-hand sides above have the same form as those appearing in the forcing (4.5).

If the characteristic $r+is$ is not a root of $p(z)$ then one needs through the d^{th} derivative of the KEY identity. These should be evaluated at $z = r + is$. A linear combination of the resulting $d + 1$ equations (and their conjugates if $s \neq 0$) can then be found so that its right-hand side equals any $f(t)$ given by (4.5). One then finds a particular solution of the form

$$
\begin{aligned}
Y_P(t) = {} & \big(A_0 t^d + A_1 t^{d-1} + \cdots + A_d\big) e^{rt} \cos(st) \\
& + \big(B_0 t^d + B_1 t^{d-1} + \cdots + B_d\big) e^{rt} \sin(st) \,,
\end{aligned}
\tag{4.7}
$$

where A_0, A_1, $\cdots$, A_d, and B_0, B_1, $\cdots$, B_d are real constants. Notice that when $s = 0$ the terms involving B_0, B_1, $\cdots$, B_d all vanish.

More generally, if the characteristic $r + is$ is a root of $p(z)$ of multiplicity m then one needs through the $(m+d)^{th}$ derivative of the KEY identity. These should be evaluated at $z = r + is$. Because $r + is$ is a root of multiplicity m, the first m of these will vanish when evaluated at $z = r + is$. A linear combination of the resulting $d + 1$ equations (and their conjugates if $s \neq 0$) can then be found so that its right-hand side equals any $f(t)$ given by (4.5). One then finds a particular solution of the form

$$\begin{aligned} Y_P(t) = &\big(A_0 t^{m+d} + A_1 t^{m+d-1} + \cdots + A_d t^m\big) e^{rt} \cos(st) \\ &+ \big(B_0 t^{m+d} + B_1 t^{m+d-1} + \cdots + B_d t^m\big) e^{rt} \sin(st)\,, \end{aligned} \tag{4.8}$$

where A_0, A_1, $\cdots$, A_d, and B_0, B_1, $\cdots$, B_d are constants. Notice that when $s = 0$ the terms involving B_0, B_1, $\cdots$, B_d all vanish. This case includes the previous one if we understand "$r + is$ is a root of $p(z)$ of multiplicity 0" to mean that it is not a root of $p(z)$. When one then sets $m = 0$ in (4.8), it reduces to (4.7).

Given a nonhomogeneous problem $\mathrm{L}y = f(t)$ in which the forcing $f(t)$ has the form (4.5) with degree d and characteristic $r + is$ that is a root of $p(z)$ of multiplicity m, the method of undetermined coefficients will find $Y_P(t)$ in the form (4.8) with A_0, A_1, $\cdots$, A_d, and B_0, B_1, $\cdots$, B_d as unknowns to be determined. These are the "undetermined coefficients" of the method. There are $2d + 2$ unknowns when $s \neq 0$, and only $d + 1$ unknowns when $s = 0$ because in that case the terms involving B_0, B_1, $\cdots$, B_d vanish. These unknowns can be determined in one of two ways.

1. **Direct Substitution.** You can substitute the form (4.8) directly into equation (4.3), collect like terms, and match the coefficients in front of each of the linearly independent functions in (4.5). In general this leads to a linear algebraic system of either $2d + 2$ equations (if $s \neq 0$) or $d+1$ equations (if $s = 0$) that must be solved. This is the only approach presented in the book.
2. **KEY Identity Evaluations.** You can evaluate the m^{th} through the $(m + d)^{th}$ derivative of the KEY identity at $z = r + is$, then find a linear combination of the resulting $d + 1$ equations (and their conjugates if $s \neq 0$) whose right-hand side equals any $f(t)$ given by (4.5). This is the approach presented most often in the lectures.

Both of these approaches always work. They are both fairly painless when m and d are both small and $s = 0$. When m is not small then the first approach is usually faster. When m and d are both small and $s \neq 0$ then the second approach is usually faster. For the problems you will face both m and d will be small, so $m + d$ will seldom be larger than 3, and more commonly be 0, 1, or 2. Both approaches will be presented in the following examples.

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = 6e^{2t}\,.$$

Solution: The characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 9 = (z+1)^2 + 3^2\,.$$

Its roots are $-1 \pm i3$. Hence,

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing is of the form (4.5) with degree $d = 0$ and characteristic $r + is = 2$. Notice that $m = 0$ because the characteristic 2 is not a root of $p(z)$.

Direct Substitution. Because $m = d = 0$ and $r + is = 2$, we see from (4.8) that Y_P has the form

$$Y_P(t) = Ae^{2t}\,.$$

Because

$$Y_P'(t) = 2Ae^{2t}\,, \qquad Y_P''(t) = 4Ae^{2t}\,,$$

we see that

$$\begin{aligned} \mathrm{L}Y_P(t) &= Y_P''(t) + 2Y_P'(t) + 10Y_P(t) \\ &= 4Ae^{2t} + 4Ae^{2t} + 10Ae^{2t} = 18Ae^{2t}\,. \end{aligned}$$

If we set $\mathrm{L}Y_P(t) = 6e^{2t}$ then we see that $18A = 6$, whereby $A = \frac{1}{3}$. Hence,

$$Y_P(t) = \tfrac{1}{3}e^{2t}\,.$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{1}{3}e^{2t}\,.$$

KEY Identity Evaluations. Because $m + d = 0$, we will only need the KEY identity:

$$\mathrm{L}\big(e^{zt}\big) = (z^2 + 2z + 10)e^{zt}\,.$$

Evaluate this at $z = 2$ to obtain

$$\mathrm{L}\big(e^{2t}\big) = (4 + 4 + 10)e^{2t} = 18e^{2t}\,.$$

Dividing this by 3 gives

$$\mathrm{L}\big(\tfrac{1}{3}e^{2t}\big) = 6e^{2t}\,,$$

from which we read off that

$$Y_P(t) = \tfrac{1}{3}e^{2t}\,.$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{1}{3}e^{2t}\,.$$

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = 4te^{2t} \,.$$

Solution: As before the characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2 \,.$$

Its roots are $-1 \pm i3$. Hence,

$$Y_H(t) = c_1 e^{-t} \cos(3t) + c_2 e^{-t} \sin(3t) \,.$$

To find a particular solution, first notice that the forcing is of the form (4.5) with degree $d = 1$ and characteristic $r + is = 2$. Notice that $m = 0$ because the characteristic 2 is not a root of $p(z)$.

Direct Substitution. Because $m = 0$, $d = 1$ and $r + is = 2$, we see from (4.8) that Y_P has the form

$$Y_P(t) = (A_0 t + A_1) e^{2t} \,.$$

Because

$$Y_P'(t) = 2(A_0 t + A_1)e^{2t} + A_0 e^{2t} \,, \qquad Y_P''(t) = 4(A_0 t + A_1)e^{2t} + 4A_0 e^{2t} \,,$$

we see that

$$\begin{aligned} \mathrm{L}Y_P(t) &= Y_P''(t) + 2Y_P'(t) + 10Y_P(t) \\ &= 4(A_0 t + A_1)e^{2t} + 4A_0 e^{2t} + 4(A_0 t + A_1)e^{2t} + 2A_0 e^{2t} + 10(A_0 t + A_1)e^{2t} \\ &= 18(A_0 t + A_1)e^{2t} + 6A_0 e^{2t} \\ &= 18A_0 te^{2t} + (18A_1 + 6A_0)e^{2t} \,. \end{aligned}$$

If we set $\mathrm{L}Y_P(t) = 4te^{2t}$ then by equating the coefficients of the linearly independent functions te^{2t} and e^{2t} we see that

$$18A_0 = 4 \,, \qquad 18A_1 + 6A_0 = 0 \,.$$

Upon solving this linear algebraic system for A_0 and A_1 we first find that $A_0 = \frac{2}{9}$ and then that $A_1 = -\frac{1}{3}A_0 = -\frac{2}{27}$. Hence,

$$Y_P(t) = \tfrac{2}{9} te^{2t} - \tfrac{2}{27} e^{2t} \,.$$

A general solution is therefore

$$y = c_1 e^{-t} \cos(3t) + c_2 e^{-t} \sin(3t) + \tfrac{2}{9} te^{2t} - \tfrac{2}{27} e^{2t} \,.$$

KEY Identity Evaluations. Because $m + d = 1$, we will only need the KEY identity and its first derivative with respect to z:

$$\begin{aligned} \mathrm{L}\big(e^{zt}\big) &= (z^2 + 2z + 10)\, e^{zt} \,, \\ \mathrm{L}\big(t\, e^{zt}\big) &= (z^2 + 2z + 10)\, t\, e^{zt} + (2z + 2)\, e^{zt} \,. \end{aligned}$$

Evaluate these at $z = 2$ to obtain

$$\mathrm{L}\big(e^{2t}\big) = 18e^{2t} \,, \qquad \mathrm{L}\big(t\, e^{2t}\big) = 18\, t\, e^{2t} + 6\, e^{2t} \,.$$

Because we want to isolate the $t\, e^{2t}$ term on the right-hand side, subtract one-third the first equation from the second to get

$$\mathrm{L}\big(t\, e^{2t} - \tfrac{1}{3}e^{2t}\big) = \mathrm{L}\big(t\, e^{2t}\big) - \tfrac{1}{3}\mathrm{L}\big(e^{2t}\big) = 18\, t\, e^{2t} \,.$$

After multiplying this by $\frac{2}{9}$ you can read off that

$$Y_P(t) = \tfrac{2}{9} t e^{2t} - \tfrac{2}{27} e^{2t} \,.$$

A general solution is therefore

$$y = c_1 e^{-t} \cos(3t) + c_2 e^{-t} \sin(3t) + \tfrac{2}{9} t e^{2t} - \tfrac{2}{27} e^{2t} \,.$$

We now illustrate an alternative way to use the KEY Identity Evaluations approach when you have more than one evaluation of the KEY identity and its derivatives, such as in the previous example.

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = 4te^{2t} \,.$$

Alternative KEY Identity Evaluations: Proceed as in the last example up to the point

$$\mathrm{L}\big(e^{2t}\big) = 18e^{2t} \,, \qquad \mathrm{L}\big(t\, e^{2t}\big) = 18\, t\, e^{2t} + 6\, e^{2t} \,.$$

If we set $Y_P(t) = A_0\, t\, e^{2t} + A_1\, e^{2t}$ then we see that

$$\begin{aligned} \mathrm{L}Y_P(t) = A_0 \mathrm{L}\big(t\, e^{2t}\big) + A_1 \mathrm{L}\big(e^{2t}\big) &= A_0\big(18\, t\, e^{2t} + 6\, e^{2t}\big) + A_1 18 e^{2t} \\ &= 18 A_0\, t\, e^{2t} + (6A_0 + 18A_1) e^{2t} \,. \end{aligned}$$

If we set $\mathrm{L}Y_P(t) = 4te^{2t}$ then by equating the coefficients of the linearly independent functions $t\, e^{2t}$ and e^{2t} we see that

$$18A_0 - 4 \,, \qquad 6A_0 + 18A_1 = 0 \,.$$

Upon solving this linear algebraic system for A_0 and A_1 we first find that $A_0 = \frac{2}{9}$ and then that $A_1 = -\frac{1}{3}A_0 = -\frac{2}{27}$. Hence,

$$Y_P(t) = \tfrac{2}{9}te^{2t} - \tfrac{2}{27}e^{2t}\,.$$

A general solution is therefore

$$y = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t) + \tfrac{2}{9}te^{2t} - \tfrac{2}{27}e^{2t}\,.$$

Remark: Notice that this alternative way to using KEY Identity Evaluations led to the same linear algebraic system for A_0 and A_1 that we got for the Direct Substitution approach. This will generally be the case because they are just two different ways to evaluate $\mathrm{L}Y_P(t)$ for the same family of $Y_P(t)$.

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2y + 2\mathrm{D}y + 10y = \cos(2t)\,.$$

Solution: As before, the characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2\,.$$

Its roots are $-1 \pm i3$. Hence,

$$Y_H(t) = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing is of the form (4.5) with degree $d = 0$ and characteristic $r + is = i2$. Notice that $m = 0$ because the characteristic $i2$ is not a root of $p(z)$.

Direct Substitution. Because $m = d = 0$ and $r + is = i2$, we see from (4.8) that Y_P has the form

$$Y_P(t) = A\cos(2t) + B\sin(2t)\,.$$

Because

$$Y_P'(t) = -2A\sin(2t) + 2B\cos(2t)\,, \qquad Y_P''(t) = -4A\cos(2t) - 4B\sin(2t)\,,$$

we see that

$$\begin{aligned}
\mathrm{L}Y_P(t) &= Y_P''(t) + 2Y_P'(t) + 10Y_P(t) \\
&= -4A\cos(2t) - 4B\sin(2t) - 4A\sin(2t) + 4B\cos(2t) + 10A\cos(2t) + 10B\sin(2t) \\
&= (6A + 4B)\cos(2t) + (6B - 4A)\sin(2t)\,.
\end{aligned}$$

If we set $\mathrm{L}Y_P(t) = \cos(2t)$ then by equating the coefficients of the linearly independent functions $\cos(2t)$ and $\sin(2t)$ we see that

$$6A + 4B = 1\,, \qquad -4A + 6B = 0\,.$$

Upon solving this system we find that $A = \frac{3}{26}$ and $B = \frac{1}{13}$, whereby

$$Y_P(t) = \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t)\,.$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t)\,.$$

KEY Identity Evaluations. Because $m + d = 0$, we will only need the KEY identity:

$$\mathrm{L}\big(e^{zt}\big) = (z^2 + 2z + 10)e^{zt}\,.$$

Evaluate this at $z = i2$ to obtain

$$\mathrm{L}\big(e^{i2t}\big) = (-4 + i4 + 10)e^{i2t} = (6 + i4)e^{i2t}\,.$$

Dividing this by $6 + i4$ gives

$$\mathrm{L}\left(\frac{1}{6+i4}\,e^{i2t}\right) = e^{i2t} = \cos(2t) + i\sin(2t)\,.$$

Taking the real part of each side gives

$$\mathrm{L}\left(\mathrm{Re}\left(\frac{1}{6+i4}\,e^{i2t}\right)\right) = \cos(2t)\,,$$

from which we read off that

$$\begin{aligned} Y_P(t) &= \mathrm{Re}\left(\frac{1}{6+i4}\,e^{i2t}\right) = \mathrm{Re}\left(\frac{6-i4}{6^2+4^2}\,e^{i2t}\right) = \tfrac{1}{52}\,\mathrm{Re}\big((6-i4)e^{i2t}\big) \\ &= \tfrac{1}{52}\,\mathrm{Re}\big((6-i4)(\cos(2t) + i\sin(2t))\big) = \tfrac{6}{52}\cos(2t) + \tfrac{4}{52}\sin(2t)\,. \end{aligned}$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t)\,.$$

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 4y = t\,\cos(2t)\,.$$

Solution: This problem has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 4 = z^2 + 2^2\,.$$

Its roots are $\pm i2$. Hence,

$$Y_H(t) = c_1 \cos(2t) + c_2 \sin(2t)\,.$$

To find a particular solution, first notice that the forcing is of the form (4.5) with degree $d = 1$ and characteristic $r+ = i2$. Notice that $m = 1$ because the characteristic $i2$ is a root of $p(z)$.

KEY Identity Evaluations. Because $m + d = 2$, we will need the first two derivatives of the KEY identity:

$$\begin{aligned}
\mathrm{L}\big(e^{zt}\big) &= (z^2+4)e^{zt}\,,\\
\mathrm{L}\big(t\,e^{zt}\big) &= (z^2+4)t\,e^{zt} + 2z\,e^{zt}\,,\\
\mathrm{L}\big(t^2e^{zt}\big) &= (z^2+4)t^2e^{zt} + 4z\,t\,e^{zt} + 2e^{zt}\,.
\end{aligned}$$

Evaluate these at $z = i2$ to obtain

$$\mathrm{L}\big(e^{i2t}\big) = 0\,, \qquad \mathrm{L}\big(t\,e^{i2t}\big) = i4\,e^{i2t}\,, \qquad \mathrm{L}\big(t^2e^{i2t}\big) = i8\,t\,e^{i2t} + 2e^{i2t}\,.$$

Because $t\,\cos(2t) = \mathrm{Re}(t\,e^{i2t})$, we want to isolate the $t\,e^{i2t}$ term on the right-hand side. This is done by multiplying the second equation by $i\frac{1}{2}$ and adding it to the third to find

$$\mathrm{L}\big((t^2 + i\tfrac{1}{2}t)e^{i2t}\big) = \mathrm{L}\big(t^2e^{i2t}\big) + i\tfrac{1}{2}\mathrm{L}\big(t\,e^{i2t}\big) = i8\,t\,e^{i2t}\,.$$

Now divide this by $i8$ to obtain

$$\mathrm{L}\left(\frac{t^2 + i\frac{1}{2}t}{i8}\,e^{i2t}\right) = t\,e^{i2t}\,,$$

from which we read off that

$$Y_P(t) = \mathrm{Re}\left(\frac{t^2 + i\frac{1}{2}t}{i8}\,e^{i2t}\right) = \frac{t}{16}\,\mathrm{Re}\big((1 - i2t)e^{i2t}\big) = \frac{t}{16}\big(\cos(2t) + 2t\sin(2t)\big)\,.$$

A general solution is therefore

$$y = c_1\cos(2t) + c_2\sin(2t) + \tfrac{1}{16}t\,\cos(2t) + \tfrac{1}{8}t^2\sin(2t)\,.$$

Direct Substitution. Because $m = d = 1$ and $r + is = i2$, we see from (4.8) that Y_P has the form

$$Y_P(t) = (A_0t^2 + A_1t)\cos(2t) + (B_0t^2 + B_1t)\sin(2t)\,.$$

Because

$$\begin{aligned}
Y_P'(t) &= -2(A_0t^2 + A_1t)\sin(2t) + (2A_0t + A_1)\cos(2t) \\
&\quad + 2(B_0t^2 + B_1t)\cos(2t) + (2B_0t + B_1)\sin(2t) \\
&= \big(2B_0t^2 + 2(B_1 + A_0)t + A_1\big)\cos(2t) - \big(2A_0t^2 + 2(A_1 - B_0)t - B_1\big)\sin(2t)\,, \\
Y_P''(t) &= -2\big(2B_0t^2 + 2(B_1 + A_0)t + A_1\big)\sin(2t) + \big(4B_0t + 2(B_1 + A_0)\big)\cos(2t) \\
&\quad - 2\big(2A_0t^2 + 2(A_1 - B_0)t - B_1\big)\cos(2t) - \big(4A_0t + 2(A_1 - B_0)\big)\sin(2t) \\
&= -\big(4A_0t^2 + (4A_1 - 8B_0)t - 4B_1 - 2A_0\big)\cos(2t) \\
&\quad - \big(4B_0t^2 + (4B_1 + 8A_0)t + 4A_1 - 2B_0\big)\sin(2t)\,,
\end{aligned}$$

we see that

$$\begin{aligned}
\mathrm{L}Y_P(t) &= Y_P''(t) + 4Y_P(t) \\
&= -\Big[\big(4A_0t^2 + (4A_1 - 8B_0)t - 4B_1 - 2A_0\big)\cos(2t) \\
&\qquad + \big(4B_0t^2 + (4B_1 + 8A_0)t + 4A_1 - 2B_0\big)\sin(2t)\Big] \\
&\quad + 4\Big[(A_0t^2 + A_1t)\cos(2t) + (B_0t^2 + B_1t)\sin(2t)\Big] \\
&= \big(8B_0t + 4B_1 + 2A_0\big)\cos(2t) - \big(8A_0t + 4A_1 - 2B_0\big)\sin(2t)\,.
\end{aligned}$$

If we set $\mathrm{L}Y_P(t) = t\,\cos(2t)$ then by equating the coefficients of the linearly independent functions $\cos(2t)$, $t\,\cos(2t)$, $\sin(2t)$, and $t\,\sin(2t)$, we see that

$$4B_1 + 2A_0 = 0\,, \qquad 8B_0 = 1\,, \qquad 4A_1 - 2B_0 = 0\,, \qquad 8A_0 = 0\,.$$

The solution of this system is $A_0 = 0$, $B_0 = \frac{1}{8}$, $A_1 = \frac{1}{16}$, and $B_1 = 0$, whereby

$$Y_P(t) = \tfrac{1}{16}t\,\cos(2t) + \tfrac{1}{8}t^2\sin(2t)\,.$$

A general solution is therefore

$$y = c_1\cos(2t) + c_2\sin(2t) + \tfrac{1}{16}t\,\cos(2t) + \tfrac{1}{8}t^2\sin(2t)\,.$$

Remark: The above example is typical of a case when the KEY identity evaluations approach is far faster than direct substitution. This is because the forcing has a positive degree $d = 1$ and a conjugate pair characteristic, $r \pm is = \pm i2$, of small multiplicity, $m = 1$. This advantage would be much more dramatic for larger d, but would diminish some for larger m.

The method of undetermined coefficients can be applied multiple times to construct a particular solution of $\mathrm{L}y = f(t)$ whenever

(1) the differential operator L has constant coefficients,

(2) the forcing $f(t)$ is a sum of terms of the form (4.5), each with different characteristics.

The first of these conditions is always easy to verify by inspection. Verification of the second usually can also be done by inspection, but sometimes it might require the use of a trigonometric or some other identity. You should be able identify when a forcing $f(t)$ can be expressed as a sum of terms of the form (4.5), and when it is, to read-off the degree and characteristic of each component.

Example: The forcing of the equation $\mathrm{L}y = \cos(t)^2$ can be written as a sum of terms of the form (4.5) by using the identity $\cos(t)^2 = (1 + \cos(2t))/2$. One sees that

$$\mathrm{L}y = \cos(t)^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos(2t)\,.$$

The both terms on the right-hand side above have the form (4.5); the first with degree $d = 0$ and characteristic $r + is = 0$, and the second with degree $d = 0$ and characteristic $r + is = i2$.

Example: The forcing of the equation $\mathrm{L}y = \sin(2t)\cos(3t)$ can be written as a sum of terms of the form (4.5) by using the identity

$$\sin(2t)\cos(3t) = \tfrac{1}{2}\big(\sin(3t + 2t) - \sin(3t - 2t)\big) = \tfrac{1}{2}\big(\sin(5t) - \sin(t)\big)\,.$$

One sees that

$$\mathrm{L}y = \sin(2t)\cos(3t) = \tfrac{1}{2}\sin(5t) - \tfrac{1}{2}\sin(t)\,.$$

The both terms on the right-hand side above have the form (4.5); the first with degree $d = 0$ and characteristic $r + is = i5$, and the second with degree $d = 0$ and characteristic $r + is = i$.

Example: The forcing of the equation $\mathrm{L}y = \tan(t)$ cannot be written as a sum of term of the form (4.5) because every such function is smooth (infinitely differentiable) while $\tan(t)$ is not defined at $t = \frac{\pi}{2} + m\pi$ for every integer m.

Given a nonhomogeneous problem $\mathrm{L}y = f(t)$ in which the forcing $f(t)$ is a sum of terms that each have the form (4.5), you must first identify the characteristic of each term and group all the terms with the same characteristic together. You then decompose $f(t)$ as

$$f(t) = f_1(t) + f_2(t) + \cdots + f_g(t)\,,$$

where each $f_j(t)$ contains all the terms of a given characteristic. Each $f_j(t)$ will then have the form (4.5) for some degree d and some characteristic $r + is$. You then can apply the method of undetermined coefficients to find particular solutions Y_{jP} to each of

$$\mathrm{L}Y_{1P}(t) = f_1(t)\,, \quad \mathrm{L}Y_{2P}(t) = f_2(t)\,, \quad \cdots \quad \mathrm{L}Y_{gP}(t) = f_g(t)\,. \tag{4.9}$$

Then $Y_P(t) = Y_{1P}(t) + Y_{2P}(t) + \cdots + Y_{gP}(t)$ is a particular solution of $\mathrm{L}y = f(t)$.

Example: If $\mathrm{L}y = \mathrm{D}^4 y + 25\mathrm{D}^2 y = f(t)$ with

$$f(t) = e^{2t} + 9\cos(5t) + 4t^2 e^{2t} - 7t\sin(5t) + 8 - 6t\,,$$

you decompose $f(t)$ as $f(t) = f_1(t) + f_2(t) + f_3(t)$, where

$$f_1(t) = 8 - 6t\,, \quad f_2(t) = \big(1 + 4t^2\big)e^{2t}\,, \quad f_3(t) = 9\cos(5t) - 7t\sin(5t)\,.$$

Here $f_1(t)$, $f_2(t)$, and $f_3(t)$ contain all the terms of $f(t)$ with characteristic 0, 2, and $i5$, respectively. They each have the form (4.5) with degree 1, 2, and 1 respectively. The characteristic polynomial is $p(z) = z^4 + 25z^2 = z^2(z^2 + 5^2)$, which has roots 0, 0, $-i5$, $i5$. We thereby see that the characteristics 0, 2, and $i5$ have multiplicities 2, 0, and 1 respectively. We can then read off from (4.8) that the method of undertermined coefficients will yield particular solutions for each of the problems in (4.9) that have the forms

$$\begin{aligned}
Y_{1P}(t) &= A_0 t^3 + A_1 t^2\,,\\
Y_{2P}(t) &= \big(A_0 t^2 + A_1 t + A_2\big)e^{2t}\,,\\
Y_{3P}(t) &= \big(A_0 t^2 + A_1 t\big)\cos(5t) + \big(B_0 t^2 + B_1 t\big)\sin(5t)\,.
\end{aligned}$$

The KEY identity approach is usually the fastest way to evaluate the undertermined coefficients in such problems because the KEY identity and its derivatives only have to be computed once. In the problem at hand, $m + d$ for the characteristics 0, 2, and $i5$ are 3, 2, and 2. We therefore need the KEY identity and its first three derivatives:

$$\begin{aligned}
\mathrm{L}\big(e^{zt}\big) &= \big(z^4 + 25z^2\big)\,e^{zt}\,,\\
\mathrm{L}\big(t\,e^{zt}\big) &= \big(z^4 + 25z^2\big)\,t\,e^{zt} + \big(4z^3 + 50z\big)\,e^{zt}\,,\\
\mathrm{L}\big(t^2 e^{zt}\big) &= \big(z^4 + 25z^2\big)\,t^2 e^{zt} + 2\big(4z^3 + 50z\big)\,t\,e^{zt} + \big(12z^2 + 50\big)\,e^{zt}\,,\\
\mathrm{L}\big(t^3 e^{zt}\big) &= \big(z^4 + 25z^2\big)\,t^3 e^{zt} + 3\big(4z^3 + 50z\big)\,t^2 e^{zt} + 3\big(12z^2 + 50\big)\,t\,e^{zt} + 24z\,e^{zt}\,,
\end{aligned}$$

For the characteristic 0 one has $m = 2$ and $m + d = 3$, so we evaluate the second through third derivative of the KEY identity at $z = 0$ to obtain

$$\mathrm{L}\big(t^2\big) = 50\,, \qquad \mathrm{L}\big(t^3\big) = 150t\,.$$

It follows that $\mathrm{L}\big(\frac{2}{25}t^2 - \frac{1}{25}t^3\big) = 8 - 6t$, whereby $Y_{1P}(t) = \frac{2}{25}t^2 - \frac{1}{25}t^3$.

For the characteristic 2 one has $m = 0$ and $m + d = 2$, so we evaluate the zeroth through second derivative of the KEY identity at $z = 2$ to obtain

$$\begin{aligned}
\mathrm{L}\big(e^{2t}\big) &= 116\,e^{2t}\,,\\
\mathrm{L}\big(t\,e^{2t}\big) &= 116\,t\,e^{2t} + 132e^{2t}\,,\\
\mathrm{L}\big(t^2 e^{2t}\big) &= 116\,t^2 e^{2t} + 264\,t\,e^{2t} + 98\,e^{2t}\,.
\end{aligned}$$

You eliminate $t\,e^{2t}$ from the right-hand sides by multiplying the second equation by $\frac{264}{116}$ and subtracting it from the third equation, thereby obtaining

$$\mathrm{L}\big(t^2e^{2t} - \tfrac{264}{116}t\,e^{2t}\big) = 116\,t^2e^{2t} + \big(98 - \tfrac{264}{116}132\big)\,e^{2t}\,.$$

Dividing this by 29 gives

$$\mathrm{L}\big(\tfrac{1}{29}t^2e^{2t} - \tfrac{66}{29^2}t\,e^{2t}\big) = 4\,t^2e^{2t} + \big(\tfrac{98}{29} - \tfrac{66\cdot 132}{29^2}\big)\,e^{2t}\,.$$

You eliminate e^{2t} from the right-hand side above by multiplying the first equation by $\frac{1}{116}\big(\frac{98}{29} - \frac{66\cdot 132}{29^2}\big)$ and subtracting it from the above equation, thereby obtaining

$$\mathrm{L}\Big(\tfrac{1}{29}t^2e^{2t} - \tfrac{66}{29^2}t\,e^{2t} - \tfrac{1}{116}\big(\tfrac{98}{29} - \tfrac{66\cdot 132}{29^2}\big)\,e^{2t}\Big) = 4\,t^2e^{2t}\,.$$

Next, by multiplying the first equation by $\frac{1}{116}$ and adding it to the above equation you obtain

$$\mathrm{L}\Big(\tfrac{1}{29}t^2e^{2t} - \tfrac{66}{29^2}t\,e^{2t} - \tfrac{1}{116}\big(\tfrac{98}{29} - \tfrac{66\cdot 132}{29^2} - 1\big)\,e^{2t}\Big) = \big(1 + 4\,t^2\big)\,e^{2t}\,,$$

whereby $Y_{2P}(t) = \frac{1}{29}t^2e^{2t} - \frac{66}{29^2}t\,e^{2t} - \frac{1}{116}\big(\frac{98}{29} - \frac{66\cdot 132}{29^2} - 1\big)\,e^{2t}$.

For the characteristic $i5$ one has $m = 1$ and $m+d = 2$, so we evaluate the first through second derivative of the KEY identity at $z = i5$ to obtain

$$\mathrm{L}\big(t\,e^{i5t}\big) = -i250\,e^{i5t}\,,\qquad \mathrm{L}\big(t^2e^{i5t}\big) = -i2\cdot 250\,t\,e^{i5t} - 250\,e^{i5t}\,.$$

Upon multiplying the first equation by i and adding it to the second we find that

$$\mathrm{L}\big(t^2e^{i5t} + it\,e^{i5t}\big) = -i2\cdot 250\,t\,e^{i5t}\,.$$

The first equation and the above equation imply

$$\mathrm{L}\big(i\tfrac{1}{250}t\,e^{i5t}\big) = e^{i5t}\,,\qquad \mathrm{L}\big(\tfrac{1}{500}t^2e^{i5t} + i\tfrac{1}{500}t\,e^{i5t}\big) = -i\,t\,e^{i5t}\,.$$

The real parts of the above equations are

$$\mathrm{L}\big(-\tfrac{1}{250}t\,\sin(5t)\big) = \cos(5t)\,,\qquad \mathrm{L}\big(\tfrac{1}{500}t^2\cos(5t) - \tfrac{1}{500}t\,\sin(5t)\big) = t\,\sin(5t)\,.$$

This implies that

$$\mathrm{L}\big(-\tfrac{9}{250}t\,\sin(5t) - \tfrac{7}{500}t^2\cos(5t) + \tfrac{7}{500}t\,\sin(5t)\big) = 9\cos(5t) - 7t\,\sin(5t)\,,$$

whereby $Y_{3P}(t) = -\frac{11}{500}t\,\sin(5t) - \frac{7}{500}t^2\cos(5t)$.

4.3: Green Functions: Constant Coefficient Case. This method can be used to construct a particular solution of an n^{th} order nonhomogeneous linear ODE in the normal form

$$\mathrm{L}y = f(t) \tag{4.10}$$

whenever the differential operator L has constant coefficients,

$$\mathrm{L} = \mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n\,. \tag{4.11}$$

Specifically, a particular solution of (4.10) is given by

$$Y_P(t) = \int_{t_I}^{t} g(t-s)f(s)\,\mathrm{d}s\,, \tag{4.12}$$

where t_I is any initial time and $g(t)$ is the solution of the homogeneous initial-value problem

$$\mathrm{L}g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 0\,, \quad \cdots \quad g^{(n-2)}(0) = 0\,, \quad g^{(n-1)}(0) = 1\,. \tag{4.13}$$

The function g is called the *Green function* associated with the operator L. Solving the initial-value problem (4.13) for the Green function is never difficult. The method thereby reduces the problem of finding a particular solution $Y_P(t)$ for any forcing $f(t)$ to that of evaluating the integral in (4.12). However, evaluating this integral explicitly can be quite difficult or impossible. At worst, you can leave your answer in terms of a definite integral.

Before we verify that $Y_P(t)$ given by (4.12) is a solution of (4.10), let us work a few examples to show how the method works.

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y - y = \frac{2}{e^t + e^{-t}}\,.$$

Solution: The operator L has constant coefficients. Its characteristic polynomial is given by $p(z) = z^2 - 1 = (z-1)(z+1)$, which has roots ± 1. A general solution of the associated homogeneous equation is therefore

$$Y_H(t) = c_1 e^t + c_2 e^{-t}\,.$$

By (4.13) the Green function g associated with L is the solution of the initial-value problem

$$\mathrm{D}^2 g - g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

Set $g(t) = c_1 e^t + c_2 e^{-t}$. The first initial condition implies $g(0) = c_1 + c_2 = 0$. Because $g'(t) = c_1 e^t - c_2 e^{-t}$, the second condition implies $g'(0) = c_1 - c_2 = 1$. Upon solving these equations you find that $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. The Green function is therefore $g(t) = \frac{1}{2}(e^t - e^{-t}) = \sinh(t)$. The particular solution given by (4.12) with $t_I = 0$ is then

$$Y_P(t) = \int_0^t \frac{e^{t-s} - e^{-t+s}}{e^s + e^{-s}}\,\mathrm{d}s = e^t \int_0^t \frac{e^{-s}}{e^s + e^{-s}}\,\mathrm{d}s - e^{-t} \int_0^t \frac{e^s}{e^s + e^{-s}}\,\mathrm{d}s\,.$$

The definite integrals in the above expression can be evaluated as

$$\begin{aligned}
\int_0^t \frac{e^{-s}}{e^s + e^{-s}}\,\mathrm{d}s &= \int_0^t \frac{e^{-2s}}{1 + e^{-2s}}\,\mathrm{d}s = -\tfrac{1}{2}\log\big(1 + e^{-2s}\big)\Big|_{s=0}^{t} = -\tfrac{1}{2}\log\left(\frac{1 + e^{-2t}}{2}\right),\\
\int_0^t \frac{e^{s}}{e^s + e^{-s}}\,\mathrm{d}s &= \int_0^t \frac{e^{2s}}{e^{2s} + 1}\,\mathrm{d}s = \tfrac{1}{2}\log\big(e^{2s} + 1\big)\Big|_{s=0}^{t} = \tfrac{1}{2}\log\left(\frac{e^{2t} + 1}{2}\right).
\end{aligned}$$

The above expression for $Y_P(t)$ thereby becomes

$$Y_P(t) = -\tfrac{1}{2}e^t \log\left(\frac{1 + e^{-2t}}{2}\right) - \tfrac{1}{2}e^{-t}\log\left(\frac{e^{2t} + 1}{2}\right).$$

A general solution is therefore $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ and $Y_P(t)$ are given above.

Remark: Notice that in the above example the definite integral in the expression for $Y_P(t)$ given by (4.12) splits into two definite integrals over s whose integrands do not involve t. This kind of splitting always happens. In general, if L is an n^{th} order operator then the expression for $Y_P(t)$ given by (4.12) always splits into n definite integrals over s whose integrands do not involve t.

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 9y = \frac{27}{16 + 9\sin(3t)^2}\,.$$

Solution: The operator L has constant coefficients. Its characteristic polynomial is given by $p(z) = z^2 + 9 = z^2 + 3^2$, which has roots $\pm i3$. A general solution of the associated homogeneous equation is therefore

$$Y_H(t) = c_1\cos(3t) + c_2\sin(3t)\,.$$

By (4.13) the Green function g associated with L is the solution of the initial-value problem

$$\mathrm{D}^2 g + 9g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

Set $g(t) = c_1\cos(3t) + c_2\sin(3t)$. The first initial condition implies $g(0) = c_1 = 0$, whereby $g(t) = c_2\sin(3t)$. Because $g'(t) = 3c_2\cos(2t)$, the second condition implies $g'(0) = 3c_2 = 1$, whereby $c_2 = \frac{1}{3}$. The Green function is therefore $g(t) = \frac{1}{3}\sin(3t)$. The particular solution given by (4.12) with $t_I = 0$ is then

$$Y_P(t) = \int_0^t \sin\big(3(t-s)\big)\,\frac{9}{16 + 9\sin(3s)^2}\,\mathrm{d}s\,.$$

Because $\sin\big(3(t-s)\big) = \sin(3t)\cos(3s) - \cos(3t)\sin(3s)$, you can express $Y_P(t)$ as

$$Y_P(t) = \sin(3t)\int_0^t \frac{9\cos(3s)}{16 + 9\sin(3s)^2}\,\mathrm{d}s - \cos(3t)\int_0^t \frac{9\sin(3s)}{16 + 9\sin(3s)^2}\,\mathrm{d}s\,.$$

The definite integrals in the above expression can be evaluated as

$$\begin{aligned}
\int_0^t \frac{9\cos(3s)}{16 + 9\sin(3s)^2}\,\mathrm{d}s &= \int_0^t \frac{\frac{9}{16}\cos(3s)}{1 + \frac{9}{16}\sin(3s)^2}\,\mathrm{d}s \\
&= \tfrac{1}{4}\tan^{-1}\big(\tfrac{3}{4}\sin(3s)\big)\Big|_{s=0}^{t} = \tfrac{1}{4}\tan^{-1}\big(\tfrac{3}{4}\sin(3t)\big)\,. \\
\int_0^t \frac{9\sin(3s)}{16 + 9\sin(3s)^2}\,\mathrm{d}s &= \int_0^t \frac{9\sin(3s)}{25 - 9\cos(3s)^2}\,\mathrm{d}s = \int_0^t \frac{\frac{9}{25}\sin(3s)}{1 - \frac{9}{25}\cos(3s)^2}\,\mathrm{d}s \\
&= -\tfrac{1}{10}\log\left(\frac{1 + \frac{3}{5}\cos(3s)}{1 - \frac{3}{5}\cos(3s)}\right)\Bigg|_{s=0}^{t} = -\tfrac{1}{10}\log\left(\frac{1 + \frac{3}{5}\cos(3t)}{1 - \frac{3}{5}\cos(3t)}\,\frac{\frac{2}{5}}{\frac{8}{5}}\right).
\end{aligned}$$

Here the first integral has the form

$$\tfrac{1}{4}\int \frac{\mathrm{d}u}{1 + u^2} = \tfrac{1}{4}\tan^{-1}(u) + C\,, \qquad \text{where } u = \tfrac{3}{4}\sin(3s)\,,$$

while by using partial fractions you see that the second has the form

$$-\tfrac{1}{5}\int \frac{\mathrm{d}u}{1 - u^2} = -\tfrac{1}{10}\log\left(\frac{1+u}{1-u}\right) + C\,, \qquad \text{where } u = \tfrac{3}{5}\cos(3s)\,.$$

The above expression for $Y_P(t)$ thereby becomes

$$Y_P(t) = \tfrac{1}{4}\sin(3t)\tan^{-1}\big(\tfrac{3}{4}\sin(3t)\big) + \tfrac{1}{10}\sin(3t)\log\left(\frac{5 + 3\cos(3t)}{5 - 3\cos(3t)}\,\frac{1}{4}\right).$$

A general solution is therefore $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ and $Y_P(t)$ are given above.

Remark: One can evaluate any integral whose integrand is a rational function of sine and cosine. The integrals in the above example are of this type. The next example illustrates what happens in most instances when the Green function method is applied — namely, the integrals that arise cannot be evaluated analytically.

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 5y = \frac{1}{1+t^2}\,.$$

Solution: The operator L has constant coefficients. Its characteristic polynomial is given by $p(z) = z^2 + 2z + 5 = (z+1)^2 + 2^2$, which has roots $-1 \pm i2$. A general solution of the associated homogeneous equation is therefore

$$Y_H(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)\,.$$

By (4.13) the Green function g associated with L is the solution of the initial-value problem

$$\mathrm{D}^2 g + 2\mathrm{D}g + 5g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

Set $g(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$. The first initial condition implies $g(0) = c_1 = 0$, whereby $g(t) = c_2 e^{-t}\sin(2t)$. Because $g'(t) = 2c_2 e^{-t}\cos(2t) - c_2 e^{-t}\sin(2t)$, the second condition implies $g'(0) = 2c_2 = 1$, whereby $c_2 = \frac{1}{2}$. The Green function is therefore $g(t) = \frac{1}{2}e^{-t}\sin(2t)$. The particular solution given by (4.12) with $t_I = \pi$ is then

$$Y_P(t) = \int_\pi^t \tfrac{1}{2} e^{-t+s}\sin\big(2(t-s)\big)\frac{1}{1+s^2}\,\mathrm{d}s\,.$$

Because $\sin\big(2(t-s)\big) = \sin(2t)\cos(2s) - \cos(2t)\sin(2s)$, you can express $Y_P(t)$ as

$$Y_P(t) = \tfrac{1}{2}e^{-t}\sin(2t)\int_\pi^t \frac{e^s\cos(2s)}{1+s^2}\,\mathrm{d}s - \tfrac{1}{2}e^{-t}\cos(2t)\int_\pi^t \frac{e^s\sin(2s)}{1+s^2}\,\mathrm{d}s\,.$$

The above definite integrals cannot be evaluated analytically. You can therefore leave the answer in terms of these integrals. A general solution is therefore $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ and $Y_P(t)$ are given above.

Remark: One should never use the Green function method when the method of undetermined coefficients can be applied. For example, for the equation

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 5y = t\,,$$

the Green function method leads to the expression

$$Y_P(t) = \tfrac{1}{2}e^{-t}\sin(2t)\int_0^t e^s\cos(2s)s\,\mathrm{d}s - \tfrac{1}{2}e^{-t}\cos(2t)\int_0^t e^s\sin(2s)s\,\mathrm{d}s\,.$$

The evaluation of these integrals requires several integration by parts. The time it would take you to do this is much longer than the time it would take you to carry out the method of undetermined coefficients!

Now let us verify that $Y_P(t)$ given by (4.12) indeed always gives a solution of (4.10) when $g(t)$ is the solution of the initial-value problem (4.13). We will use the fact from multivariable calculus that for any continuously differentiable $K(t,s)$ one has

$$\mathrm{D}\int_{t_I}^{t} K(t,s)\,\mathrm{d}s = K(t,t) + \int_{t_I}^{t} \partial_t K(t,s)\,\mathrm{d}s\,, \qquad \text{where} \quad \mathrm{D} = \frac{d}{dt}\,.$$

Because $g(0) = 0$, you see from (4.12) that

$$\mathrm{D}Y_P(t) = g(0)f(t) + \int_{t_I}^{t} \mathrm{D}g(t-s)f(s)\,\mathrm{d}s = \int_{t_I}^{t} \mathrm{D}g(t-s)f(s)\,\mathrm{d}s\,.$$

If $2 < n$ then because $\mathrm{D}g(0) = g'(0) = 0$, you see from the above that

$$\mathrm{D}^2Y_P(t) = g'(0)f(t) + \int_{t_I}^{t} \mathrm{D}^2g(t-s)f(s)\,\mathrm{d}s = \int_{t_I}^{t} \mathrm{D}^2g(t-s)f(s)\,\mathrm{d}s\,.$$

If you continue to argue this way then because $\mathrm{D}^{k-1}g(0) = g^{(k-1)}(0) = 0$ for $k < n$, you see that for every $k < n$

$$\mathrm{D}^kY_P(t) = g^{(k-1)}(0)f(t) + \int_{t_I}^{t} \mathrm{D}^kg(t-s)f(s)\,\mathrm{d}s = \int_{t_I}^{t} \mathrm{D}^kg(t-s)f(s)\,\mathrm{d}s\,.$$

Similarly, because $\mathrm{D}^{n-1}g(0) = g^{(n-1)}(0) = 1$ then you see that

$$\mathrm{D}^nY_P(t) = g^{(n-1)}(0)f(t) + \int_{t_I}^{t} \mathrm{D}^ng(t-s)f(s)\,\mathrm{d}s = f(t) + \int_{t_I}^{t} \mathrm{D}^ng(t-s)f(s)\,\mathrm{d}s\,.$$

Because $\mathrm{L}g(t) = 0$, it follows that $\mathrm{L}g(t-s) = 0$. Then by the above formulas for $\mathrm{D}^kY_P(t)$ you see that

$$\begin{aligned}
\mathrm{L}Y_P(t) = p(\mathrm{D})Y_P(t) &= \mathrm{D}^nY_P(t) + a_1\mathrm{D}^{n-1}Y_P(t) + \cdots + a_{n-1}\mathrm{D}Y_P(t) + a_nY_P(t) \\
&= f(t) + \int_{t_I}^{t} \mathrm{D}^ng(t-s)f(s)\,\mathrm{d}s + \int_{t_I}^{t} a_1\mathrm{D}^{n-1}g(t-s)f(s)\,\mathrm{d}s \\
&\quad + \cdots + \int_{t_I}^{t} a_{n-1}\mathrm{D}g(t-s)f(s)\,\mathrm{d}s + \int_{t_I}^{t} a_ng(t-s)f(s)\,\mathrm{d}s \\
&= f(t) + \int_{t_I}^{t} p(\mathrm{D})g(t-s)f(s)\,\mathrm{d}s \\
&= f(t) + \int_{t_I}^{t} \mathrm{L}g(t-s)f(s)\,\mathrm{d}s = f(t)\,.
\end{aligned}$$

Therefore, $Y_P(t)$ given by (4.12) is a solution of (4.10). Moreover, one sees from the above calculations that it is the unique solution of (4.10) that satisfies the initial conditions

$$Y_P(t_I) = 0\,, \quad Y_P'(t_I) = 0\,, \quad \cdots \quad Y_P^{(n-1)}(t_I) = 0\,.$$

5. Mechanical Vibrations

5.1. Spring-Mass Systems. Consider a spring hanging from a support. When an object of mass m is attached to the free end of the spring, the object will eventually come to rest at a lower position. Let y_o and y_r be the vertical rest positions of the free end of the spring without and with the mass attached. We will assume that the mass is constrained to only move vertically and want to describe the vertical postition $y(t)$ of the mass as a function of time t when the mass is initially displaced from y_r, or is given some initial velocity, or is driven by an external force $F_{ext}(t)$.

The forces acting on the mass that we will consider are the gravitational force F_{grav}, the spring force F_{spr}, the damping or drag force F_{damp}, and the external or driving force F_{ext}. Newton's law of motion then states that

$$m\frac{d^2y}{dt^2} = F_{grav} + F_{spr} + F_{damp} + F_{ext}\,. \tag{5.1}$$

Always be sure you are working in one of the standard systems of units. In MKS units length is given in meters (m), time in seconds (sec), mass in kilograms (kg), and force in Newtons (1 Newton = 1 kg m/sec^2). In CGS units length is given in centimeters (cm), time in seconds (sec), mass in grams (g), and force in dynes (1 dyne = 1 g cm/sec^2). In British units length is given in feet (ft), time in seconds (sec), mass in slugs (sl), and force in pounds (1 pound = 1 sl ft/sec^2).

The gravitational force F_{grav} is simply the downward weight of the mass. If we assume a uniform gravitational acceleration g then

$$F_{grav} = -mg\,, \tag{5.2}$$

where $g = 980$ cm/sec^2 in CGS units, $g = 9.8$ m/sec^2 in MKS units, and $g = 32$ ft/sec^2 in British units.

The spring force is modeled by Hooke's law

$$F_{spr} = -k(y - y_o)\,, \tag{5.3}$$

where k is the so-called spring constant or spring coefficient. This is a fairly good model provided $y - y_o$ does not get too big. When there is no external driving force, the mass has a rest position $y_r < y_o$ that satisfies

$$0 = F_{grav} + F_{spr} \quad \text{at } y = y_r\,.$$

Hence, we have

$$mg = -k(y_r - y_o) = k(y_o - y_r) = k|y_r - y_o|\,. \tag{5.4}$$

Sometimes you will be given $|y_r - y_o|$ and have to figure out k from this relation.

The damping force is modeled by

$$F_{damp} = -\gamma \frac{dy}{dt}, \tag{5.5}$$

where $\gamma \geq 0$ is the so-called damping coefficient. This is not as good a model for damping force as Hooke's Law was for the spring force, but we will use it because of its simplicity. Sometimes you will be given $|F_{damp}|$ at a particular speed and have to determine γ from this relation.

If we place (5.2), (5.3), and (5.5) into Newton's law of motion (5.1) and neglect the external driving, we obtain

$$m\frac{d^2y}{dt^2} + \gamma \frac{dy}{dt} + ky = ky_o - mg\,.$$

We see from (5.4) that $ky_o - mg = ky_r$, whereby

$$m\frac{d^2y}{dt^2} + \gamma \frac{dy}{dt} + ky = ky_r\,.$$

This clearly has the particular solution $y = y_r$. If we let $y(t) = y_r + h(t)$ then $h(t)$ satisfies the homogeneous equation

$$m\frac{d^2h}{dt^2} + \gamma \frac{dh}{dt} + kh = 0\,.$$

Here $h(t)$ is simply the displacement of the mass from its rest position y_r. If the external driving is present, this becomes

$$m\frac{d^2h}{dt^2} + \gamma \frac{dh}{dt} + kh = F_{ext}(t)\,. \tag{5.6}$$

We will study the motion of this spring-mass system building up its complexity from simplest case.

5.2. Unforced, Undamped Motion ($F_{ext} = 0$, $\gamma = 0$). In this case (5.6) reduces to

$$m\frac{d^2h}{dt^2} + kh = 0\,,$$

or in normal form

$$\frac{d^2h}{dt^2} + \frac{k}{m}\,h = 0\,. \tag{5.7}$$

Its characteristic polynomial is

$$p(z) = z^2 + \frac{k}{m}\,,$$

which has roots $\pm i\omega_o$ where

$$\omega_o = \sqrt{\frac{k}{m}}\,. \tag{5.8}$$

A general solution of equation (5.7) is

$$h(t) = c_1 \cos(\omega_o t) + c_2 \sin(\omega_o t)\,. \tag{5.9}$$

For the initial condtions $h(0) = h_0$ and $h'(0) = h_1$ this becomes

$$h(t) = h_0 \cos(\omega_o t) + h_1 \frac{\sin(\omega_o t)}{\omega_o}\,.$$

Such motion is called *simple harmonic motion.* It is oscillitory motion with a single frequency ω_o.

Because ω_o is associated with the spring constant k through (5.8), it is called the *natural frequency* of the spring. The associated *natural period* T_o is therefore

$$T_o = \frac{2\pi}{\omega_o}\,.$$

In CGS, MKS, and British units ω_o is given in radians/sec, or simply 1/sec because radians are considered to be nondimensional. Then T_o is given in sec.

The simple harmonic motion (5.9) is nontrivial whenever either c_1 or c_2 is nonzero. In that case we can express it in the so-called applitude-phase form

$$h(t) = A \cos(\omega_o t - \delta)\,,$$

where $A > 0$ is its amplitude and δ in $[0, 2\pi)$ is its phase. By the cosine addition formula the above form can be expanded as

$$h(t) = A \cos(\delta) \cos(\omega_o t) + A \sin(\delta) \sin(\omega_o t)\,.$$

Upon comparing this with (5.9) we see that

$$A \cos(\delta) = c_1\,, \qquad A \sin(\delta) = c_2\,.$$

This shows that (A, δ) are simply the polar coordinates of the point in the plane whose cartesian coordinates are (c_1, c_2). Clearly $A = \sqrt{c_1^2 + c_2^2} > 0$ while δ satisfies

$$\cos(\delta) = \frac{c_1}{A}\,, \qquad \sin(\delta) = \frac{c_2}{A}\,.$$

There is a unique δ in $[0, 2\pi)$ that satisfies these equations.

Example: A mass of 10 grams stretches a spring 5 cm when at rest. At $t = 0$ the mass is set in motion from its rest position with a downward velocity of 35 cm/sec. Neglect damping and external forces.

a) What is the displacement of the mass as a function of time?
b) What is the amplitude, phase, frequency, and period of the motion?
c) At what positive time does the mass first return to its rest position?

Solution: Because $g = 980$ cm/sec^2, we can find k by setting

$$k \cdot 5 = mg = 10 \cdot 980 \quad \text{dynes}\,,$$

whereby

$$k = \frac{10 \cdot 980}{5} \quad \text{dynes/cm}\,.$$

Because we are neglecting damping and external forces, the equation of motion takes the form

$$m\,\frac{d^2h}{dt^2} + k\,h = 0\,,$$

which becomes

$$10\,\frac{d^2h}{dt^2} + \frac{10 \cdot 980}{5}\,h = 0\,.$$

Bringing this into normal form gives

$$\frac{d^2h}{dt^2} + \frac{980}{5}\,h = 0\,,$$

which becomes

$$\frac{d^2h}{dt^2} + 196\,h = 0\,.$$

Because $\omega_o^2 = 196$, one sees that $\omega_o = 14$ 1/sec.

A general solution of the equation of motion is therefore

$$h(t) = c_1 \cos(14t) + c_2 \sin(14t)\,.$$

The initial conditions are $h(0) = 0$ and $h'(0) = -35$ cm/sec. Because

$$h'(t) = -14c_1 \sin(14t) + 14c_2 \cos(14t)\,,$$

the boundary conditions imply that

$$h(0) = c_1 = 0\,, \qquad h'(0) = 14c_2 = -35\,,$$

which implies $c_1 = 0$ and $c_2 = -\frac{5}{2}$. From this you can read off the following.

a) The displacement of the mass as a function of time is

$$h(t) = -\tfrac{5}{2}\sin(14t) = \tfrac{5}{2}\cos(14t - \tfrac{3\pi}{2}) \quad \text{cm}\,.$$

b) The amplitude of the motion is $\frac{5}{2}$ cm, the phase is $\frac{3\pi}{2}$, the frequency is 14 1/sec, and the period is $\frac{\pi}{7}$ sec.

c) The positive time at which the mass first returns to its rest position is $t = \frac{\pi}{14}$.

5.3. Unforced, Damped Motion ($F_{ext} = 0$, $\gamma > 0$). In this case (5.6) reduces to

$$m\frac{d^2h}{dt^2} + \gamma\frac{dh}{dt} + kh = 0\,,$$

or in normal form

$$\frac{d^2h}{dt^2} + \frac{\gamma}{m}\frac{dh}{dt} + \frac{k}{m}\,h = 0\,. \tag{5.10}$$

Its characteristic polynomial is

$$p(z) = z^2 + \frac{\gamma}{m}\,z + \frac{k}{m}\,.$$

If we complete the square this has the form

$$p(z) = (z+\mu)^2 + \omega_o^2 - \mu^2\,. \tag{5.11}$$

where the positive constants μ and ω_o are defined by

$$\mu = \frac{\gamma}{2m}\,, \qquad \omega_o = \sqrt{\frac{k}{m}}\,.$$

It is clear there are three cases to consider.

- When $0 < \mu < \omega_o$ there is a conjugate pair of roots $-\mu \pm i\nu$ where

$$\nu = \sqrt{\omega_o^2 - \mu^2}\,. \tag{5.12}$$

- When $\mu = \omega_o$ there is a real double root $-\mu$, $-\mu$.
- When $\mu > \omega_o$ there is two simple real roots $-\mu \pm \sqrt{\mu^2 - \omega_o^2}$.

These are called the under damped, critically damped, and over damped cases respectively.

For the *under damped* case a general solution is

$$h(t) = c_1 e^{-\mu t}\cos(\nu t) + c_2 e^{-\mu t}\sin(\nu t)\,.$$

Whenever either c_1 or c_2 is nonzero this can be put into the amplitude-phase form

$$h(t) = Ae^{-\mu t}\cos(\nu t - \delta)\,,$$

where $A = \sqrt{c_1^2 + c_2^2} > 0$ and $0 \le \delta < 2\pi$ satisfies

$$\cos(\delta) = \frac{c_1}{A}\,, \qquad \sin(\delta) = \frac{c_2}{A}\,.$$

The displacement is therefore an exponentially decaying simple harmonic motion with the time-dependent amplitude $Ae^{-\mu t}$, frequency ν, and phase δ. In this context ν given by (5.12) is called the *quasi frequency* of the system and the associated period $2\pi/\nu$ is called the *quasi period*. Notice that

$$\nu < \omega_o \,, \qquad \frac{2\pi}{\nu} > T_o \,.$$

In other words, the quasi frequency is always less than the natural frequency, while the quasi period is always greater than the natural period.

For the *critically damped* case a general solution is

$$h(t) = c_1 e^{-\mu t} + c_2 t\, e^{-\mu t} \,.$$

The displacement therefore has at most one zero and decays like $t\, e^{-\mu t}$ whenever $c_2 \neq 0$.

For the *over damped* case a general solution is

$$h(t) = c_1 e^{-\mu_+ t} + c_1 e^{-\mu_- t} \,,$$

where

$$\mu_\pm = \mu \pm \sqrt{\mu^2 - \omega_o^2} \,. \tag{5.13}$$

Notice that $0 < \mu_- < \mu < \mu_+$. The displacement therefore has at most one zero and decays like $t\, e^{-\mu_- t}$ whenever $c_2 \neq 0$. Because $\mu_- < \mu$ one sees that in this case the decay of the displacement is slower than in either the under or critically damped cases.

Remark: This damped spring system is a good model for shock absorbers in a car. When the shock absorbers are over damped one gets a jarring ride, while when they are under damped one gets a bouncy ride. Shock absorbers are tuned to be critically damped, which gives the least jarring and least bouncy ride.

Remark: The spring system is said to be extremely over damped when μ is much greater than ω_o. In that case we can use the approximation

$$\sqrt{\mu^2 - \omega_o^2} = \mu \sqrt{1 - \frac{\omega_o^2}{\mu^2}} \approx \mu \left(1 - \frac{\omega_o^2}{2\mu^2} \right) = \mu - \frac{\omega_o^2}{2\mu} \,,$$

to approximate μ_- and μ_+ by

$$\mu_- \approx \frac{\omega_o^2}{2\mu} \,, \qquad \mu_+ \approx 2\mu - \frac{\omega_o^2}{2\mu} \,.$$

In this regime these decay rates are very different from each other and from μ, with

$$\frac{\mu_-}{\mu} \approx \frac{\mu}{\mu_+} \approx \frac{\omega_o^2}{2\mu^2} \,,$$

5.4. Forced, Undamped Motion ($F_{ext} \neq 0$, $\gamma = 0$). In this case (5.6) reduces to

$$m\frac{d^2h}{dt^2} + kh = F_{ext}(t)\,.$$

We will study external forces of the form

$$F_{ext}(t) = F\cos(\omega t)\,.$$

The equation then has the normal form

$$\frac{d^2h}{dt^2} + \omega_o^2 h = a\cos(\omega t)\,, \tag{5.14}$$

where the natural frequency ω_o and the driving acceleration a are given by

$$\omega_o = \sqrt{\frac{k}{m}}\,, \qquad a = \frac{F}{m}\,.$$

Equation (5.14) may be solved by the method of undetermined coefficients. The characteristic polynomial is $p(z) = z^2 + \omega_o^2$, which has roots $\pm i\omega_o$. The forcing has characteristic $\pm i\omega$, which has multiplicity 0 when $\omega \neq \omega_o$ and multiplicity 1 when $\omega = \omega_o \neq 0$.

For $\omega \neq \omega_o$, if we impose the inital conditions

$$h(0) = h_0\,, \quad \text{and} \quad h'(0) = h_1\,,$$

then the solution is found to be

$$h(t;\omega) = h_0\cos(\omega_o t) + h_1\frac{\sin(\omega_o t)}{\omega_o} + \frac{a}{\omega_o^2 - \omega^2}\big(\cos(\omega t) - \cos(\omega_o t)\big)\,. \tag{5.15}$$

This is not simple harmonic motion because more than one frequency in involved. Such motion is sometimes called poly harmonic.

For $\omega = \omega_o \neq 0$, if we impose the inital conditions

$$h(0) = h_0\,, \quad \text{and} \quad h'(0) = h_1\,,$$

then the solution is found to be

$$h(t;\omega_o) = h_0\cos(\omega_o t) + h_1\frac{\sin(\omega_o t)}{\omega_o} + \frac{a}{2\omega_o}\,t\,\sin(\omega_o t)\,. \tag{5.16}$$

This is also not simple harmonic motion. In fact, its amplitude grows linearly in t! This phenomenon of *resonance* that occurs when the driving frequency ω becomes equal to the natural frequency ω_o of the system. Because l'Hopital's rule implies

$$\lim_{\omega\to\omega_o}\frac{\cos(\omega t) - \cos(\omega_o t)}{\omega_o^2 - \omega^2} = \lim_{\omega\to\omega_o}\frac{-t\,\sin(\omega t)}{-2\omega} = \frac{t\,\sin(\omega_o t)}{2\omega_o}\,,$$

we see that formula (5.16) is what you obtain by taking the limit $\omega \to \omega_o$ in formula (5.15).

You can understand the onset of resonance as $\omega \to \omega_o$ by using the identity

$$\cos(\omega t) - \cos(\omega_o t) = -2\sin\left(\frac{\omega - \omega_o}{2}\,t\right)\sin\left(\frac{\omega + \omega_o}{2}\,t\right),$$

to re-express formula (5.15) as

$$h(t;\omega) = h_0\cos(\omega_o t) + h_1\frac{\sin(\omega_o t)}{\omega_o} + A(t)\,\sin\left(\frac{\omega + \omega_o}{2}\,t\right),$$

where

$$A(t) = \frac{2a}{\omega^2 - \omega_o^2}\,\sin\left(\frac{\omega - \omega_o}{2}\,t\right).$$

When $\omega - \omega_o$ is very small compared to ω and ω_o then $A(t)$ will be a very slowly varying function of t compared to $\sin\big((\omega+\omega_o)t/2\big)$. In that case $\sin\big((\omega+\omega_o)t/2\big)$ will oscillate very many times during a period over which $A(t)$ oscillates just once. These rapid oscillations will have an amplitude of $|A(t)|$, which slowly oscillates between 0 and $2a/(\omega^2 - \omega_o^2)$. This slow oscillation is the phenomenon of beating.