HIGHER-ORDER LINEAR
ORDINARY DIFFERENTIAL EQUATIONS IV:
Variable Coefficient Nonhomogeneous Case

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Because the presentation of this material in class will differ from that in the book, I felt that notes that closely follow the class presentation might be appreciated.

7. Variable Coefficient Nonhomogeneous Case

7.1: Introduction. We now return to study nonhomogeneous linear equations for the general case of with variable coefficients that was begun in Section 4.1. An n^{th} order nonhomogeneous linear ODE has the normal form

$$\mathrm{L}(t)y = f(t)\,, \tag{7.1}$$

where the differential operator $\mathrm{L}(t)$ has the normal form

$$\mathrm{L}(t) = \frac{d^n}{dt^n} + a_1(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{d}{dt} + a_n(t)\,. \tag{7.2}$$

We will assume throughout this section that the coefficients $a_1, a_2, \cdots, a_n$ and the forcing f are continuous over an interval (t_L, t_R), so that Therorem 1.1 can be applied.

Recall the following strategy for constructing general solutions of the nonhomogeneous equation (7.1) that we developed in Section 4.1.

(1) Find a general solution $Y_H(t)$ of the associated homogeneous equation $\mathrm{L}(t)y = 0$.

(2) Find a particular solution $Y_P(t)$ of equation (7.1).

(3) Then $Y_H(t) + Y_P(t)$ is a general solution of (7.1).

If you can find a fundamental set $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_n(t)$ of solutions to the associated homogeneous equation $\mathrm{L}(t)y = 0$ then a general solution of that equation is given by

$$Y_H(t) = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,.$$

In the ensuing sections we will explore two methods to construct a particular solution $Y_P(t)$ of equation (7.1) from $f(t)$ and the fundamental set $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_n(t)$. One method is called *variation of parameters*, while the other is called the *general Green function* method, which is an extension of the Green function method presented in Section 4.3 for constant coefficient equations to the case of variable coefficient equations. We will see that these methods are essentially equivalent. What lies behind them is the following.

> **Important Fact:** If you know a general solution of the associated homogeneous equation $\mathrm{L}(t)y = 0$ then you can *always* reduce the construction of a general solution of (7.1) to the problem of finding n primitives.

Because at this point you only know how to find general solutions of homogeneous equations with constant coefficients, problems you will be given will generally fall into one of two categories. Either (1) the operator $\mathrm{L}(t)$ will have variable coefficients and you will be given a fundamental set of solutions for the associated homogeneous equation, or (2) the operator $\mathrm{L}(t)$ will have constant coefficients (i.e. $\mathrm{L}(t) = \mathrm{L}$) and you will be expected to find a fundamental set of solutions for the associated homogeneous equation. In the later case the general Green function method reduces to the method presented in Section 4.3.

7.2: Variation of Parameters: Second Order Case. We begin by deriving the method of variation of parameters for second order equations that are in the normal form

$$\mathrm{L}(t)y = \frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)y = f(t)\,. \tag{7.3}$$

Suppose you know that $Y_1(t)$ and $Y_2(t)$ are linearly independent solutions of the associated homogeneous equation $\mathrm{L}(t)y = 0$. A general solution of the associated homogeneous equation is therefore given by

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t)\,. \tag{7.4}$$

The idea of the method of variation of parameters is to seek solutions of (7.3) in the form

$$y = u_1(t)Y_1(t) + u_2(t)Y_2(t)\,. \tag{7.5}$$

In other words you simply replace the arbitrary constants c_1 and c_2 in (7.5) with unknown functions $u_1(t)$ and $u_2(t)$. These functions are the varying parameters referred to in the title of the method. These two functions will be governed by a system of two equations, one of which is derived by requiring that (7.3) is satisfied, and the other of which is chosen to simplify the resulting system.

Let us see how this is done. Differentiating (7.5) yields

$$\frac{dy}{dt} = u_1(t)Y_1'(t) + u_2(t)Y_2'(t) + u_1'(t)Y_1(t) + u_2'(t)Y_2(t)\,. \tag{7.6}$$

We now choose to impose the condition

$$u_1'(t)Y_1(t) + u_2'(t)Y_2(t) = 0\,, \tag{7.7}$$

whereby (7.6) simplifies to

$$\frac{dy}{dt} = u_1(t)Y_1'(t) + u_2(t)Y_2'(t)\,. \tag{7.8}$$

Differentiating (7.8) then yields

$$\frac{d^2y}{dt^2} = u_1(t)Y_1''(t) + u_2(t)Y_2''(t) + u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t)\,. \tag{7.9}$$

Now substituting (7.5), (7.8), and (7.9) into (7.3), grouping the terms that multiply $u_1(t)$, $u_1'(t)$, $u_2(t)$, and $u_2'(t)$, and using the fact that $\mathrm{L}(t)Y_1(t) = 0$ and $\mathrm{L}(t)Y_2(t) = 0$, we obtain

$$\begin{aligned}
f(t) = \mathrm{L}(t)y &= \frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)y \\
&= \big[u_1(t)Y_1''(t) + u_2(t)Y_2''(t) + u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t)\big] \\
&\quad + a_1(t)\big[u_1(t)Y_1'(t) + u_2(t)Y_2'(t)\big] + a_2(t)\big[u_1(t)Y_1(t) + u_2(t)Y_2(t)\big] \\
&= u_1(t)\big[Y_1''(t) + a_1(t)Y_1'(t) + a_2(t)Y_1(t)\big] + u_1'(t)Y_1'(t) \\
&\quad + u_2(t)\big[Y_2''(t) + a_1(t)Y_2'(t) + a_2(t)Y_2(t)\big] + u_2'(t)Y_2'(t) \\
&= u_1(t)\big[\mathrm{L}(t)Y_1(t)\big] + u_1'(t)Y_1'(t) + u_2(t)\big[\mathrm{L}(t)Y_2(t)\big] + u_2'(t)Y_2'(t) \\
&= u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t)\,.
\end{aligned} \tag{7.10}$$

The resulting system that governs $u_1(t)$ and $u_2(t)$ is thereby given by (7.7) and (7.10):

$$\begin{aligned} u_1'(t)Y_1(t) + u_2'(t)Y_2(t) &= 0\,, \\ u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t) &= f(t)\,. \end{aligned} \tag{7.11}$$

This is a linear system of two algebraic equations for $u_1'(t)$ and $u_2'(t)$. Because

$$\big(Y_1(t)Y_2'(t) - Y_2(t)Y_1'(t)\big) = W[Y_1, Y_2](t) \neq 0\,,$$

one can always solve this system to find

$$u_1'(t) = -\frac{Y_2(t)f(t)}{W[Y_1, Y_2](t)}\,, \qquad u_2'(t) = \frac{Y_1(t)f(t)}{W[Y_1, Y_2](t)}\,,$$

or equivalently

$$u_1(t) = -\int \frac{Y_2(t)f(t)}{W[Y_1, Y_2](t)}\,\mathrm{d}t\,, \qquad u_2(t) = \int \frac{Y_1(t)f(t)}{W[Y_1, Y_2](t)}\,\mathrm{d}t\,. \tag{7.12}$$

Letting $u_{1P}(t)$ and $u_{2P}(t)$ be any primitives of the respective right-hand sides above, one sees that

$$u_1(t) = c_1 + u_{1P}(t)\,, \qquad u_2(t) = c_2 + u_{2P}(t)\,,$$

whereby (7.5) yields the general solution

$$y = c_1Y_1(t) + u_{1P}(t)Y_1(t) + c_2Y_2(t) + u_{2P}(t)Y_2(t)\,.$$

Notice that this decomposes as $y = Y_H(t) + Y_P(t)$ where

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t)\,, \qquad Y_P(t) = u_{1P}(t)Y_1(t) + u_{2P}(t)Y_2(t)\,. \tag{7.13}$$

There are two approaches to applying variation of parameters. One mentioned in the book is to memorize the formulas (7.12). I am not a fan of this approach for a couple of reasons. First, students often confuse which of the two formulas gets the minus sign. Second, and more importantly, these formulas do not cleanly generalize to the higher order case. The other approach is to construct the linear system (7.11), which can then be rather easily solved for $u_1'(t)$ and $u_2'(t)$. The work it takes to solve this system is about the same work as it takes to generate the integrands in (7.12). The linear system (7.11) is symmetric in $u_1'(t)$ and $u_2'(t)$, so is less subject to sign errors. Moreover, it also has a clean generalization to the higher order case. Whichever approach you take, you will be led to the same two integrals.

Given $Y_1(t)$ and $Y_2(t)$, a fundamental set of solutions to the associated homogeneous equation, you proceed as follows.

1) Write the form of the solution you seek:

$$y = u_1(t)Y_1(t) + u_2(t)Y_2(t)\,.$$

2) Write the linear algebraic system for $u_1'(t)$ and $u_2'(t)$:

$$\begin{aligned} u_1'(t)Y_1(t) + u_2'(t)Y_2(t) &= 0\,, \\ u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t) &= f(t)\,. \end{aligned}$$

   The form of the left-hand sides of this system mimics the form of the solution you seek. The first equation simply replaces $u_1(t)$ and $u_2(t)$ with $u_1'(t)$ and $u_2'(t)$, while the second also replaces $Y_1(t)$ and $Y_2(t)$ with $Y_1'(t)$ and $Y_2'(t)$. The $f(t)$ on the right-hand side will be correct only if you have written the equation $\mathrm{L}(t)y = f(t)$ in normal form!

3) Solve the linear algebraic system to find explicit expressions for $u_1'(t)$ and $u_2'(t)$. This is always very easy to do, especially if you start with the first equation.

4) Find primitives $u_{1P}(t)$ and $u_{2P}(t)$ of these expressions. If you cannot find a primitive analytically then express that primitive in terms of a definite integral. One then has

$$u_1(t) = c_1 + u_{1P}(t)\,, \qquad u_2(t) = c_2 + u_{2P}(t)\,,$$

   where c_1 and c_2 are the arbitrary constants of integration.

5) Upon placing this result into the form of the solution that you wrote down in step 1, you will obtain the general solution $y = Y_H(t) + Y_P(t)$, where

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t)\,, \qquad Y_P(t) = u_{1P}(t)Y_1(t) + u_{2P}(t)Y_2(t)\,.$$

   For initial-value problems you must determine c_1 and c_2 from the initial conditions.

Example: Find a general solution of

$$\frac{d^2y}{dt^2} + y = \sec(t)\,.$$

Before presenting the solution, notice that while this equation has constant coefficients, the forcing is not of the form that would allow you to use the method of undetermined coefficients. You should be able to recognize this right away. While you can use the Green function method to solve this problem, here we will solve it using variation of parameters.

Solution: Because this problem has constant coefficients, it is easily found that

$$Y_H(t) = c_1\cos(t) + c_2\sin(t)\,.$$

Hence, we will seek a solution of the form

$$y = u_1(t)\cos(t) + u_2(t)\sin(t)\,,$$

where

$$u_1'(t)\cos(t) + u_2'(t)\sin(t) = 0\,,$$
$$-u_1'(t)\sin(t) + u_2'(t)\cos(t) = \sec(t)\,.$$

Solving this system by any means you choose yields

$$u_1'(t) = -\frac{\sin(t)}{\cos(t)}\,, \qquad u_2'(t) = 1\,.$$

These can be integrated analytically to obtain

$$u_1(t) = c_1 + \log(|\cos(t)|)\,, \qquad u_2(t) = c_2 + t\,.$$

Therefore a general solution is

$$y = c_1\cos(t) + c_2\sin(t) + \log(|\cos(t)|)\cos(t) + t\sin(t)\,.$$

Remark: The primitives $u_1(t)$ and $u_2(t)$ that we had to find above are the same ones needed to evaluate the integrals that arise when you solve this problem with the Green function method. This will always be the case.

Example: Given that t and $t^2 - 1$ are a fundamental set of solutions of the associated homogeneous equation, find a general solution of

$$(1+t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = (1+t^2)^2 e^t\,.$$

Before presenting the solution, you should be able to recognize that this equation has nonconstant coefficients, and thereby see that you must use either variation of parameters or a general Green function to solve this problem. You should also notice that this equation is not in normal form, so you should bring it into the normal form

$$\frac{d^2y}{dt^2} - \frac{2t}{1+t^2}\,\frac{dy}{dt} + \frac{2}{1+t^2}\,y = (1+t^2)e^t\,.$$

Solution: Because t and $t^2 - 1$ are a fundamental set of solutions of the associated homogeneous equation, we have

$$Y_H(t) = c_1 t + c_2(t^2 - 1)\,.$$

Hence, we will seek a solution of the form

$$y = u_1(t)t + u_2(t)(t^2 - 1)\,,$$

where

$$\begin{aligned} u_1'(t)t + u_2'(t)(t^2-1) &= 0\,, \\ u_1'(t)1 + u_2'(t)2t &= (1+t^2)e^t\,. \end{aligned}$$

Solving this system by any means you choose yields

$$u_1'(t) = -(t^2-1)e^t\,, \qquad u_2'(t) = te^t\,.$$

These can be integrated analytically "by parts" to obtain

$$u_1(t) = c_1 - (t-1)^2 e^t\,, \qquad u_2(t) = c_2 + (t-1)e^t\,.$$

Therefore a general solution is

$$\begin{aligned} y &= c_1 t + c_2(t^2-1) - (t-1)^2 e^t t + (t-1)e^t(t^2-1) \\ &= c_1 t + c_2(t^2-1) + (t-1)^2 e^t\,. \end{aligned}$$

7.3: Variation of Parameters: Higher Order Case. The method of variation of parameters extends to higher order linear equations in the normal form

$$\mathrm{L}(t)y = \frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = f(t)\,. \tag{7.14}$$

While this material was not covered in class and you will not be tested on it, a summary is given here for the sake of completeness.

Suppose you know that $Y_1(t), Y_2(t), \cdots, Y_n(t)$ are linearly independent solutions of the associated homogeneous equation $\mathrm{L}(t)y = 0$. A general solution of the associated homogeneous equation is therefore given by

$$Y_H(t) = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,.$$

The idea of the method of variation of parameters is to seek solutions of (7.14) in the form

$$y = u_1(t)Y_1(t) + u_2(t)Y_2(t) + \cdots + u_n(t)Y_n(t)\,, \tag{7.15}$$

where $u_1'(t), u_2'(t), \cdots, u_n'(t)$ satisfy the linear algebraic system

$$\begin{aligned} u_1'(t)Y_1(t) \;+\; u_2'(t)Y_2(t) \;+\cdots+\; u_n'(t)Y_n(t) &= 0\,, \\ u_1'(t)Y_1'(t) \;+\; u_2'(t)Y_2'(t) \;+\cdots+\; u_n'(t)Y_n'(t) &= 0\,, \\ &\;\,\vdots \\ u_1'(t)Y_1^{(n-2)}(t) + u_2'(t)Y_2^{(n-2)}(t) + \cdots + u_n'(t)Y_n^{(n-2)}(t) &= 0\,, \\ u_1'(t)Y_1^{(n-1)}(t) + u_2'(t)Y_2^{(n-1)}(t) + \cdots + u_n'(t)Y_n^{(n-1)}(t) &= f(t)\,. \end{aligned} \tag{7.16}$$

Because

$$\det\begin{pmatrix} Y_1(t) & Y_2(t) & \cdots & Y_n(t) \\ Y_1'(t) & Y_2'(t) & \cdots & Y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ Y_1^{(n-1)}(t) & Y_2^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t) \end{pmatrix} = W[Y_1, Y_2, \cdots, Y_n](t) \neq 0\,,$$

the linear algebraic system (7.16) may be solved (by any method you choose) to find explicit expressions for $u_1'(t)$, $u_2'(t)$, $\cdots$, $u_n'(t)$. For example, when $n = 3$ you find

$$u_1'(t) = \frac{W[Y_2, Y_3](t)f(t)}{W[Y_1, Y_2, Y_3](t)}\,, \quad u_2'(t) = \frac{W[Y_3, Y_1](t)f(t)}{W[Y_1, Y_2, Y_3](t)}\,, \quad u_3'(t) = \frac{W[Y_1, Y_2](t)f(t)}{W[Y_1, Y_2, Y_3](t)}\,.$$

Find primitives $u_{1P}(t)$, $u_{2P}(t)$, $\cdots$, $u_{nP}(t)$ of these expressions. If you cannot find a primitive analytically then express that primitive in terms of a definite integral. One then has

$$u_1(t) = c_1 + u_{1P}(t)\,, \quad u_2(t) = c_2 + u_{2P}(t)\,, \quad \cdots \quad u_n(t) = c_n + u_{nP}(t)\,,$$

where $c_1, c_2, \cdots, c_n$ are the arbitrary constants of integration. The general solution given by (7.15) is therefore $y = Y_H(t) + Y_P(t)$, where

$$\begin{aligned} Y_H(t) &= c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,, \\ Y_P(t) &= u_{1P}(t)Y_1(t) + u_{2P}(t)Y_2(t) + \cdots + u_{nP}(t)Y_n(t)\,. \end{aligned}$$

For initial-value problems you must determine $c_1, c_2, \cdots, c_n$ from the initial conditions.

7.4: General Green Functions: Second Order Case. We now derive a Green function for second order equations that are in the normal form

$$\mathrm{L}(t)y = \frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)y = f(t)\,. \tag{7.17}$$

Suppose you know that $Y_1(t)$ and $Y_2(t)$ are linearly independent solutions of the associated homogeneous equation $\mathrm{L}(t)y = 0$. The starting point of our derivation will be the particular solution $Y_P(t)$ given in (7.13) — namely,

$$Y_P(t) = u_{1P}(t)Y_1(t) + u_{2P}(t)Y_2(t)\,,$$

where $u_{1P}(t)$ and $u_{2P}(t)$ are primitives that satisfy

$$u_{1P}'(t) = -\frac{Y_2(s)f(s)}{W[Y_1, Y_2](s)}\,, \qquad u_{2P}'(t) = \frac{Y_1(s)f(s)}{W[Y_1, Y_2](s)}\,.$$

If we express $u_{1P}(t)$ and $u_{2P}(t)$ as the definite integrals

$$u_{1P}(t) = -\int_{t_I}^{t} \frac{Y_2(s)f(s)}{W[Y_1, Y_2](s)}\,\mathrm{d}s\,, \qquad u_{2P}(t) = \int_{t_I}^{t} \frac{Y_1(s)f(s)}{W[Y_1, Y_2](s)}\,\mathrm{d}s\,, \tag{7.18}$$

where t_I is any initial time inside the interval (t_L, t_R), then the particular solution $Y_P(t)$ can be expressed as

$$Y_P(t) = \int_{t_I}^{t} G(t,s) f(s)\,\mathrm{d}s\,, \tag{7.19}$$

where $G(t,s)$ is given by

$$G(t,s) = \frac{Y_1(s)Y_2(t) - Y_1(t)Y_2(s)}{W[Y_1,Y_2](s)} = \frac{\det\begin{pmatrix} Y_1(s) & Y_2(s) \\ Y_1(t) & Y_2(t) \end{pmatrix}}{\det\begin{pmatrix} Y_1(s) & Y_2(s) \\ Y_1'(s) & Y_2'(s) \end{pmatrix}}\,. \tag{7.20}$$

The method thereby reduces the problem of finding a particular solution $Y_P(t)$ for any forcing $f(t)$ to that of evaluating the integral in (7.19), which by formula (7.20) is equivalent to evaluating the two definite integrals in (7.18). Of course, evaluating these integrals explicitly can be quite difficult or impossible. You may have to leave your answer in terms of one or both of these definite integrals. Formulas (7.19) and (7.20) have natural generalizations to higher order equations with variable coefficients.

We will see that (7.19) is an extension of the Green function formula (4.12) from Section 4.3 to second order equations with variable coefficients. As with that formula, (7.19) generates the unique particular solution $Y_P(t)$ of (7.17) that satifies the initial conditions

$$Y_P(t_I) = 0\,, \qquad Y_P'(t_I) = 0\,. \tag{7.21}$$

We therefore call $G(t,s)$ the Green function for the operator $\mathrm{L}(t)$.

Before justifying the foregoing claims, let us illustrate how to construct and use this Green function.

Example: Given that t and $t^2 - 1$ are a fundamental set of solutions of the associated homogeneous equation, find a particular solution of

$$(1+t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = (1+t^2)^2 e^t\,.$$

Solution: You should first bring this equation into its normal form

$$\frac{d^2y}{dt^2} - \frac{2t}{1+t^2}\,\frac{dy}{dt} + \frac{2}{1+t^2}\,y = (1+t^2)e^t\,.$$

Because t and $t^2 - 1$ are a fundamental set of solutions of the associated homogeneous equation, the Green function $G(t,s)$ is given by (7.20) as

$$G(t,s) = \frac{\det\begin{pmatrix} s & s^2-1 \\ t & t^2-1 \end{pmatrix}}{\det\begin{pmatrix} s & s^2-1 \\ 1 & 2s \end{pmatrix}} = \frac{(t^2-1)s - t(s^2-1)}{2s^2 - (s^2-1)} = \frac{(t^2-1)s - t(s^2-1)}{s^2+1}\,.$$

Then formula (7.19) with $t_I = 1$ and $f(s) = (1+s^2)e^s$ yields

$$Y_P(t) = (t^2-1)\int_1^t se^s \,\mathrm{d}s - t\int_1^t (s^2-1)e^s \,\mathrm{d}s\,.$$

Notice that the same two integrals that arose when we treated this equation by variation of parameters on page 7. As was done there, a little integration-by-parts shows that

$$\int_1^t se^s \,\mathrm{d}s = (t-1)e^t\,, \qquad \int_1^t (s^2-1)e^s \,\mathrm{d}s = (t-1)^2 e^t\,.$$

The particular solution is therefore

$$Y_P(t) = (t^2-1)(t-1)e^t - t(t-1)^2e^t = (t-1)^2e^t\,.$$

It is clear that this solution satisfies $Y_P(1) = Y_P'(1) = 0$. Had we chosen a different value for the initial time t_I we would have obtained a different particular solution $Y_P(t)$.

Next we show that formula (7.19) generates the unique particular solution $Y_P(t)$ of (7.17) that satifies the initial conditions (7.21). It is clear from (7.19) that $Y_P(t_I) = 0$. To show that $Y_P'(t_I) = 0$ we will use the fact from multivariable calculus that for any continuously differentiable $K(t,s)$ one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_I}^t K(t,s)\,\mathrm{d}s = K(t,t) + \int_{t_I}^t \partial_t K(t,s)\,\mathrm{d}s\,.$$

We see from (7.20) that $G(t,t) = 0$. Upon differentiating (7.19) with respect to t and using the above calculus fact, we see that

$$Y_P'(t) = G(t,t)f(t) + \int_{t_I}^t \partial_t G(t,s)f(s)\,\mathrm{d}s = \int_{t_I}^t \partial_t G(t,s)f(s)\,\mathrm{d}s\,.$$

It follows that $Y_P'(t_I) = 0$, thereby showing that $Y_P(t)$ satifies the initial conditions (7.21).

The Green function $G(t,s)$ is defined by (7.20) whenever t and s are both in the interval (t_L, t_R) overwhich Y_1 and Y_2 exist. At first it might seem that $G(t,s)$ must depend upon the fundamental set of solutions that is used to construct it. We now show that this is not the case. Let us fix s and consider $G(t,s)$ as a function of t. It is clear from (7.20) that $G(t,s)$ is a linear combination of $Y_1(t)$ and $Y_2(t)$. Because $Y_1(t)$ and $Y_2(t)$ are solutions of the associated homogeneous equation $\mathrm{L}(t)y = 0$, it follows that $G(t,s)$ is too — namely, that $\mathrm{L}(t)G(t,s) = 0$. It is also clear from (7.20) that $G(t,s)\big|_{t=s} = 0$. By differentiating (7.20) with respect to t we obtain

$$\partial_t G(t,s) = \frac{Y_1(s)Y_2'(t) - Y_1'(t)Y_2(s)}{W[Y_1,Y_2](s)} = \frac{\det\begin{pmatrix} Y_1(s) & Y_2(s) \\ Y_1'(t) & Y_2'(t)\end{pmatrix}}{\det\begin{pmatrix} Y_1(s) & Y_2(s) \\ Y_1'(s) & Y_2'(s)\end{pmatrix}}\,. \tag{7.22}$$

It is clear from this that $\partial_t G(t,s)\big|_{t=s} = 1$. Collecting these facts we see for every s that $G(t,s)$ as a function of t satisfies the initial-value problem

$$\mathrm{L}(t)G(t,s) = 0\,, \qquad G(t,s)\big|_{t=s} = 0\,, \quad \partial_t G(t,s)\big|_{t=s} = 1\,. \tag{7.23}$$

This is really a family of initial-value problems — one for each s in which s plays the role of the initial time. The uniqueness theorem implies that $G(t,s)$ is uniquely determined by this family of initial-value problems. Thus, $G(t,s)$ depends only upon the operator $\mathrm{L}(t)$. In particular, it does not depend upon which fundamental set of solutions, Y_1 and Y_2, was used to construct it.

When $\mathrm{L}(t)$ has constant coefficients then it is easy to check that the family of initial-value problems (7.23) is satisfied by $G(t,s) = g(t-s)$, where $g(t)$ is the Green function that was defined by the initial-value problem (4.13) in Section 4.3. Formula (7.19) thereby extends the Green function formula (4.12) from Section 4.3 to second order equations with variable coefficients.

When $\mathrm{L}(t)$ has constant coefficients the fastest way to compute the Green function is to solve the single initial-value problem (4.13) from Section 4.3. When $\mathrm{L}(t)$ has variable coefficients you first have to find a fundamental set of solutions, $Y_1(t)$ and $Y_2(t)$, to the associated homogeneous equation. You can then construct the Green function either by formula (7.20) or by solving the family of initial-value problems (7.23). The later approach goes as follows. Because $\mathrm{L}(t)G(t,s) = 0$ for every s we know that there exist $C_1(s)$ and $C_2(s)$ such that

$$G(t,s) = Y_1(t)C_1(s) + Y_2(t)C_2(s)\,. \tag{7.24}$$

The initial conditions of (7.23) then imply that

$$\begin{aligned} 0 = \quad G(t,s)\big|_{t=s} &= Y_1(s)C_1(s) + Y_2(s)C_2(s)\,, \\ 1 = \partial_t G(t,s)\big|_{t=s} &= Y_1'(s)C_1(s) + Y_2'(s)C_2(s)\,. \end{aligned}$$

The solution of this linear algebraic system is

$$C_1(s) = -\frac{Y_2(s)}{Y_1(s)Y_2'(s) - Y_1'(s)Y_2(s)}\,, \quad C_2(s) = \frac{Y_1(s)}{Y_1(s)Y_2'(s) - Y_1'(s)Y_2(s)}\,,$$

which when plugged into (7.24) yields (7.20).

Example: Given that t and $t^2 - 1$ are a fundamental set of solutions of the associated homogeneous equation, find a particular solution of

$$(1+t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = (1+t^2)^2 e^t\,.$$

Solution: You should first bring this equation into its normal form

$$\mathrm{L}(t)y = \frac{d^2y}{dt^2} - \frac{2t}{1+t^2}\,\frac{dy}{dt} + \frac{2}{1+t^2}\,y = (1+t^2)e^t\,.$$

Because t and $t^2 - 1$ are a fundamental set of solutions of the associated homogeneous equation, by (7.24) the Green function has the form

$$G(t,s) = t\,C_1(s) + (t^2-1)\,C_2(s)\,,$$

where the initial conditions of (7.23) imply

$$\begin{aligned} 0 &= \quad G(t,s)\big|_{t=s} = s\,C_1(s) + (s^2-1)\,C_2(s)\,,\\ 1 &= \partial_t G(t,s)\big|_{t=s} = 1\,C_1(s) + 2s\,C_2(s)\,. \end{aligned}$$

These may be solved to obtain

$$C_1(s) = -\frac{s^2-1}{s^2+1}\,, \qquad C_2(s) = \frac{s}{s^2+1}\,,$$

whereby

$$G(t,s) = -t\,\frac{s^2-1}{s^2+1} + (t^2-1)\,\frac{s}{s^2+1} = \frac{(t^2-1)s - t(s^2-1)}{s^2+1}\,.$$

You then compute $y_P(t)$ by formula (7.19) as before.

7.5: General Green Functions: Higher Order Case. This method can be used to construct a particular solution of an n^{th} order nonhomogeneous linear ODE in the normal form (7.1). Specifically, a particular solution of (7.1) is given by

$$Y_P(t) = \int_{t_I}^{t} G(t,s) f(s)\,\mathrm{d}s\,, \tag{7.25}$$

where t_I is an initial time and $G(t,s)$ is given by

$$G(t,s) = \frac{1}{W[Y_1, Y_2, \cdots, Y_n](s)} \det \begin{pmatrix} Y_1(s) & Y_2(s) & \cdots & Y_n(s) \\ Y_1'(s) & Y_2'(s) & \cdots & Y_n'(s) \\ \vdots & \vdots & \vdots & \vdots \\ Y_1^{(n-2)}(s) & Y_2^{(n-2)}(s) & \cdots & Y_n^{(n-2)}(s) \\ Y_1(t) & Y_2(t) & \cdots & Y_n(t) \end{pmatrix}.$$

The function G is called the *Green function* associated with the operator $\mathrm{L}(t)$. It can also be determined as the solution of the family of initial-value problems

$$\mathrm{L}(t)G(t,s) = 0\,, \qquad G(t,s)\big|_{t=s} = \cdots = \partial_t^{n-2} G(t,s)\big|_{t=s} = 0\,, \quad \partial_t^{n-1} G(t,s)\big|_{t=s} = 1\,.$$

The method thereby reduces the problem of finding a particular solution $Y_P(t)$ for any forcing $f(t)$ to that of evaluating the integral in (7.25). However, evaluating this integral explicitly can be quite difficult or impossible. At worst, you can leave your answer in terms of definite integrals.