Quiz 1 Solutions, Math 246, Professor David Levermore
Tuesday, 3 February 2009

(1) [4] For each of the following ordinary differential equations, give its order and state whether it is linear or nonlinear.

(a) $\dfrac{\mathrm{d}^3 v}{\mathrm{d}z^3} + e^v \dfrac{\mathrm{d}v}{\mathrm{d}z} + 4v = 0$; **Solution:** third-order, nonlinear.

(b) $e^x \dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4} + x\,\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sin(x) + x^2 y$. **Solution:** fourth-order, linear.

(2) [2] Give the interval of existence for the solution of the initial-value problem

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{1}{t^2 - 9}\, z = \tan(t)\,, \qquad z(2) = 5\,.$$

(You do not have to solve this equation to answer this question!)

Solution: This problem is linear in z and is already in normal form. The coefficient $1/(t^2 - 9)$ is continuous everywhere except where $t = \pm 3$, while the forcing $\tan(t)$ is continuous everywhere except where $t = \frac{\pi}{2} + n\pi$ for some integer n. You can therefore read off that the interval of existence is $(\frac{\pi}{2}, 3)$, the endpoints of which are the singularities of $\tan(t)$ and $1/(t^2 - 9)$ respectively that bracket the initial time 2.

(3) [4] Solve the initial-value problem

$$(1 + x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy + 1 + x^2\,, \qquad y(0) = 3\,.$$

Solution: This problem is linear in y. Its normal form is

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2x}{1 + x^2}\, y = 1\,.$$

An integrating factor is $e^{A(x)}$ where

$$A'(x) = -\frac{2x}{1 + x^2}\,.$$

Setting $A(x) = -\log(1 + x^2)$, we find that $e^{A(x)} = (1 + x^2)^{-1}$. The problem therefore has the intgrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{1 + x^2}\right) = \frac{1}{1 + x^2}\,.$$

Integrating both sides yields

$$\frac{y}{1 + x^2} = \tan^{-1}(x) + c\,.$$

Applying the initial condition gives

$$\frac{3}{1 + 0^2} = \tan^{-1}(0) + c\,,$$

which implies $c = 3$. The solution is therefore

$$y = (1 + x^2)\big(\tan^{-1}(x) + 3\big)\,.$$