Quiz 2 Solutions, Math 246, Professor David Levermore
Tuesday, 10 February 2009

(1) [3] Sketch the phase-line portrait for the initial-value problem

$$\frac{\mathrm{d}w}{\mathrm{d}t} = w^2(w+2)(w-3)\,, \qquad w(0) = w_o\,.$$

Identify each stationary (equilibrium) point as either stable, unstable, or semistable. (You do not have to find the solution!)

Solution: The phase-line is

$$\begin{array}{ccccccc} + & & - & & - & & + \\ \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow \\ & -2 & & 0 & & 3 & \\ & \text{stable} & & \text{semistable} & & \text{unstable} & \end{array}$$

(2) [3] In the absence of predators the population of mosquitoes in a certain area would increase at a rate proportional to its current population and its population would triple every week. There are 90,000 mosquitoes in the area initially, and predators eat 80,000 mosquitoes per week. Write down an initial-value problem that governs the population of mosquitoes in the area at any time. (You do not have to solve the initial-value problem!)

Solution: Let $M(t)$ be the number of mosquitoes at time t weeks. Tripling every week means a growth like $3^t = e^{\log(3)t}$, which implies a growth rate of $\log(3)$. The initial-value problem that M satisfies is therefore

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \log(3)M - 80,000\,, \qquad M(0) = 90,000\,.$$

(3) [4] Find the solution of the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 e^{-y}\,, \qquad y(0) = y_o \quad \text{for some } y_o \text{ in } (-\infty, \infty)\,.$$

Give its interval of existence as a function of y_o.

Solution: This equation is separable. Its separated differential form is $e^y\,\mathrm{d}y = 3t^2\,\mathrm{d}t$, whereby

$$\int e^y\,\mathrm{d}y = \int 3t^2\,dt\,.$$

Integrating both sides we find that $e^y = t^3 + c$. The initial condition then implies that $e^{y_o} = 0^3 + c$, whereby $c = e^{y_o}$. The solution is thereby given implicity by

$$e^y = t^3 + e^{y_o}\,.$$

Wherever $t^3 + e^{y_o} > 0$ this can be solved explicitly for y to obtain

$$y = \log\left(t^3 + e^{y_o}\right)\,.$$

The condition $t^3 > -e^{y_o}$ determines that the interval of existence is $(-e^{\frac{1}{3}y_o}, \infty)$.