Quiz 4 Solutions, Math 246, Professor David Levermore
Tuesday, 3 March 2009

(1) [2] What is the interval of existence for the solution to the initial-value problem

$$\frac{\mathrm{d}^4 z}{\mathrm{d}t^4} + e^t \frac{\mathrm{d}z}{\mathrm{d}t} + \sin(3t) z = \tan(t) , \quad z(2) = z'(2) = z''(2) = z'''(2) = 1 .$$

Solution: The equation is already in linear normal form. The coefficients and forcing are defined and continuous everywhere except at $t = k\pi + \frac{\pi}{2}$ for some integer k. The initial time is $t = 2$. The interval of existence is therefore $(\frac{\pi}{2}, \frac{3\pi}{2})$.

(2) [3] Compute the Wronskian of the functions $Y_1(t) = \cos(3t)$ and $Y_2(t) = \sin(3t)$. (Evaluate the determinant and simplify.)

Solution: Because $Y_1'(t) = -3\sin(3t)$ and $Y_2'(t) = 3\cos(3t)$, the Wronskian is

$$\begin{aligned} W[Y_1, Y_2](t) = \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} &= \det\begin{pmatrix} \cos(3t) & \sin(3t) \\ -3\cos(3t) & 3\cos(3t) \end{pmatrix} \\ &= 3\cos(3t)^2 + 3\sin(3t)^2 = 3 . \end{aligned}$$

(3) [3] Given that $\cos(2t)$ and $\sin(2t)$ are linearly independent solutions of $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 4y = 0$, find the solution $Y(t)$ that satisfies the initial conditions $Y(0) = 1$, $Y'(0) = 4$.

Solution: Let $Y(t) = c_1 \cos(2t) + c_2 \sin(2t)$. Then $Y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t)$. To satisfy the initial conditions one needs

$$1 = Y(0) = c_1 , \qquad 4 = Y'(0) = 2c_2 .$$

It follows that $c_1 = 1$ and $c_2 = 2$. The solution of the initial-value problem is therefore

$$Y(t) = \cos(2t) + 2\sin(2t) .$$

(4) [2] Suppose that $Y_1(t)$ and $Y_2(t)$ are solutions of the differential equation

$$y'' + a(t)y = 0 ,$$

where $a(t)$ is continuous over $(-5, 5)$. Suppose you know that $W[Y_1, Y_2](0) = 7$. What is $W[Y_1, Y_2](1)$?

Solution: Because the coefficient of y' is zero, Abel's Theorem implies that $W[Y_1, Y_2](t)$ is constant over $(-5, 5)$. Hence, $W[Y_1, Y_2](1) = W[Y_1, Y_2](0) = 7$.