Quiz 5 Solutions, Math 246, Professor David Levermore
Tuesday, 10 March 2009

(1) [5] Give a general solution of the equation

$$\frac{\mathrm{d}^3y}{\mathrm{d}t^3} + 6\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 5\frac{\mathrm{d}y}{\mathrm{d}t} = 3e^t .$$

Solution. The coefficients are constant. The characteristic polynomial is

$$p(z) = z^3 + 6z^2 + 5z = z(z+1)(z+5) ,$$

which has roots $z = 0, -1, -5$. A general solution of the associated homogeneous equation is

$$Y_H(t) = c_1 + c_2e^{-t} + c_3e^{-5t} .$$

The forcing has degree $d = 0$ and characteristic $r + is = 1$, which has multiplicity $m = 0$. Because $d + m = 0$ we only need the KEY identity

$$\mathrm{L}\big(e^{zt}\big) = (z^3 + 6z^2 + 5z)e^{zt} .$$

Evaluating this at $z = r + is = 1$ gives

$$\mathrm{L}(e^t) = (1^3 + 6 \cdot 1^2 + 5 \cdot 1)e^t = 12e^t .$$

Dividing this by 4 yields $\mathrm{L}(\frac{1}{4}e^t) = 3e^t$, which shows that $Y_P(t) = \frac{1}{4}e^t$ is a particular solution. (Alternatively, plug $Y_P(t) = Ae^t$ into the equation and determine A.) A general solution of the equation therefore is

$$y = Y_H(t) + Y_P(t) = c_1 + c_2e^{-t} + c_3e^{-5t} + \tfrac{1}{4}e^t .$$

(2) [5] Given that $\frac{1}{5}t$ is a particular solution of the differential equation, solve the initial value problem

$$\frac{\mathrm{d}^2w}{\mathrm{d}t^2} + 4\frac{\mathrm{d}w}{\mathrm{d}t} + 5w = t , \qquad w(0) = 0 , \quad w'(0) = 1 ,$$

Solution. The coefficients are constant. The characteristic polynomial is

$$p(z) = z^2 + 4z + 5 = (z+2)^2 + 1 = (z+2)^2 + 1^2 ,$$

which has roots $z = -2+i, -1-i$. A general solution of the associated homogeneous equation is

$$W_H(t) = c_1e^{-2t}\cos(t) + c_2e^{-2t}\sin(t) .$$

Because a particular solution is $W_P(t) = \frac{1}{5}t$, a general solution of the equation is

$$w = c_1e^{-2t}\cos(t) + c_2e^{-2t}\sin(t) + \tfrac{1}{5}t .$$

The initial condition $w(0) = 0$ implies that

$$0 = w(0) = c_1e^0\cos(0) + c_2e^0\sin(0) + \tfrac{1}{5}0 = c_1 .$$

Hence, $c_1 = 0$. Then because

$$w' = c_2e^{-2t}\cos(t) - 2c_2e^{-2t}\sin(t) + \tfrac{1}{5} ,$$

the initial condition $w'(0) = 1$ implies that

$$1 = c_2e^0\cos(0) - 2c_2e^0\sin(0) + \tfrac{1}{5} = c_2 + \tfrac{1}{5} .$$

Hence, $c_2 = \frac{4}{5}$ and the solution is $w = \frac{4}{5}e^{-2t}\sin(t) + \frac{1}{5}t$.