Quiz 6 Solutions, Math 246, Professor David Levermore
Tuesday, 24 March 2009

(1) [4] Compute the Green function associated with the differential operator $\mathrm{L} = \mathrm{D}^2 + 16$.

Solution: The Green function $g(t)$ satisfies the initial-value problem

$$g'' + 16g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

The characteristic polynomial is $p(z) = z^2 + 16$, which has roots $\pm i4$. Therefore $g(t)$ has the form

$$g(t) = c_1 \cos(4t) + c_2 \sin(4t)\,.$$

Because $g(0) = c_1 \cos(0) + c_2 \sin(0) = c_1$, the first initial condition shows $c_1 = 0$. Then $g'(t) = 4c_2 \cos(4t)$. Because $g'(0) = 4c_2 \cos(0) = 4c_2$, the second initial condition shows $4c_2 = 1$, whereby $c_2 = \frac{1}{4}$. The Green function $g(t)$ is therefore given by

$$g(t) = \tfrac{1}{4} \sin(4t)\,.$$

(2) [2] The displacement of a spring-mass system is given by

$$h(t) = \sqrt{3} \cos(2t) - \sin(2t)\,.$$

What is the amplitude of this oscillation?

Solution: The amplitude-phase form of the displacement is

$$h(t) = A \cos(2t - \delta) = A \cos(\delta) \cos(2t) + A \sin(\delta) \sin(2t)\,.$$

Comparing this with the given form shows that $A \cos(\delta) = \sqrt{3}$ and $A \sin(\delta) = -1$. The applitude A is therefore given by

$$A = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2\,.$$

Remark: The phase δ is given by $\delta = \frac{11}{6}\pi$.

(3) [4] A sping-mass system is governed by the initial-value problem

$$h'' + 6h' + 25h = 0\,, \qquad h(0) = 1\,, \quad h'(0) = 0\,.$$

(a) Determine the natural frequency and period of the spring.
(b) Determine if the system in undamped, underdamped, critically damped, or overdamped.

Solution (a): Because the equation is in normal form the natural frequency ω_o is given by

$$\omega_o = \sqrt{25} = 5\,.$$

The natural period is therefore $T_o = 2\pi/\omega_o = \frac{2}{5}\pi$.

Solution (b): The characteristic polynomial is

$$p(z) = z^2 + 6z + 25 = (z+3)^2 + 16 = (z+3)^2 + 4^2\,.$$

Its roots are $-3 \pm i4$, which are a conjugate pair. The system is therefore *underdamped*.

Remark: Because the system is underdamped, its quasifrequency is given by $\nu = 4$ while its quasiperiod is given by $2\pi/\nu = \frac{1}{2}\pi$.