Quiz 7 Solutions, Math 246, Professor David Levermore
Tuesday, 31 March 2009

Short Table: $\mathcal{L}[\sin(bt)](s) = \dfrac{b}{s^2+b^2}$ for $s>0\,,$ $\mathcal{L}[e^{at}](s) = \dfrac{1}{s-a}$ for $s>a\,.$

(1) [3] Use the definition of the Laplace transform to compute $\mathcal{L}[f](s)$ for $f(t) = u(t-2)$, where u is the unit step function.

Solution: By the definitions of the Laplace transform and the unit step function

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,\mathrm{d}t = \lim_{T\to\infty} \int_2^T e^{-st}\,\mathrm{d}t\,.$$

The above limit diverges for $s \le 0$ because then $e^{-st} \ge 1$. For $s > 0$

$$\int_2^T e^{-st}\,\mathrm{d}t = -\frac{e^{-st}}{s}\bigg|_2^T = \frac{e^{-2s}}{s} - \frac{e^{-sT}}{s}\,,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\left[\frac{e^{-2s}}{s} - \frac{e^{-sT}}{s}\right] = \frac{e^{-2s}}{s} \quad \text{for } s>0\,.$$

(2) [4] Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$y'' - 4y = \sin(t)\,,\qquad y(0) = 0\,,\quad y'(0) = 3\,.\qquad$ DO NOT solve for $y(t)$, just $Y(s)$!

Solution: The Laplace transform of the initial-value problem and item 1 in the table at the top of the page with $b=1$ gives

$$\mathcal{L}[y''](s) - 4\mathcal{L}[y](s) = \mathcal{L}[\sin(t)](s) = \frac{1}{s^2+1^2} = \frac{1}{s^2+1}\,,$$

where

$$\begin{aligned}
\mathcal{L}[y](s) &= Y(s)\,,\\
\mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s)\,,\\
\mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 3\,.
\end{aligned}$$

Hence,

$$(s^2-4)Y(s) - 3 = \frac{1}{s^2+1}\,, \qquad \Longrightarrow \qquad Y(s) = \frac{1}{s^2-4}\left(3 + \frac{1}{s^2+1}\right).$$

(3) [3] Find the inverse Laplace transform $y(t)$ of the function $Y(s) = \dfrac{3s+1}{s^2-s-6}$.

Solution: A partial fractions identity gives

$$F(s) = \frac{3s+1}{s^2-s-6} = \frac{3s+1}{(s-3)(s+2)} = \frac{2}{s-3} + \frac{1}{s+2}\,.$$

Item 2 in the table at the top of the page with $a=3$ and with $a=-2$ then gives

$$\mathcal{L}^{-1}[F](t) = 2\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = 2e^{3t} + e^{-2t}\,.$$