Quiz 8 Solutions, Math 246, Professor David Levermore
Tuesday, 14 April 2009

(1) [3] Transform the equation $\dfrac{\mathrm{d}^3 v}{\mathrm{d}t^3} - 5t^4 \dfrac{\mathrm{d}^2 v}{\mathrm{d}t^2} + e^t v = \cos(3t)$ into a first-order system of ordinary differential equations.

Solution: Because the equation is third order, the first order system must have dimension three. The simplest such first order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \cos(3t) - e^t x_1 + 5t^4 x_3 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} v \\ v' \\ v'' \end{pmatrix}.$$

(2) [3] Consider two interconnected tanks filled with brine (salt water). The first tank contains 75 liters and the second contains 125 liters. Well stirred brine flows from the first tank to the second at a rate of 4 liters per hour, and from the second to the first at the same rate. At $t = 0$ there are 60 grams of salt in the first tank and 45 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution: Let $S_1(t)$ be the grams of salt in the first tank and $S_2(t)$ be the grams of salt in the second tank. These are governed by the initial-value problem

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = \frac{S_2}{125}\,4 - \frac{S_1}{75}\,4, \qquad S_1(0) = 60,$$
$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = \frac{S_1}{75}\,4 - \frac{S_2}{125}\,4, \qquad S_2(0) = 45.$$

(3) [4] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^3+1 \\ t \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 1 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

Solution.

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} t^3+1 & t^2 \\ t & 1 \end{pmatrix} = (t^3+1)\cdot 1 - t \cdot t^2 = 1.$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$.

Solution. Let $\mathbf{\Psi}(t) = \begin{pmatrix} t^3+1 & t^2 \\ t & 1 \end{pmatrix}$. Because $\dfrac{\mathrm{d}\mathbf{\Psi}}{\mathrm{d}t}(t) = \mathbf{A}(t)\mathbf{\Psi}(t)$, one has

$$\begin{aligned} \mathbf{A}(t) &= \frac{\mathrm{d}\mathbf{\Psi}}{\mathrm{d}t}(t)\,\mathbf{\Psi}(t)^{-1} = \begin{pmatrix} 3t^2 & 2t \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t^3+1 & t^2 \\ t & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 3t^2 & 2t \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -t^2 \\ -t & t^3+1 \end{pmatrix} = \begin{pmatrix} t^2 & 2t - t^4 \\ 1 & -t^2 \end{pmatrix}. \end{aligned}$$