Quiz 9, Math 246, Professor David Levermore
Tuesday, 21 April 2009

(1) [2] Given that $e^{t\mathbf{A}} = \begin{pmatrix} \cosh(2t) & \sinh(2t) \\ \sinh(2t) & \cosh(2t) \end{pmatrix}$,

solve the initial-value problem $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}\,, \quad \mathbf{x}(0) = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$.

Solution.

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0) = \begin{pmatrix} \cosh(2t) & \sinh(2t) \\ \sinh(2t) & \cosh(2t) \end{pmatrix} \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -3\cosh(2t) + 5\sinh(2t) \\ -3\sinh(2t) + 5\cosh(2t) \end{pmatrix}.$$

(2) [4] Let $\mathbf{A} = \begin{pmatrix} 6 & 3 \\ 3 & -2 \end{pmatrix}$. Compute $e^{t\mathbf{A}}$.

Solution. The characteristic polynomial is $p(z) = z^2 - 4z - 21 = (z-2)^2 - 5^2$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^{2t}\left[\mathbf{I}\cosh(5t) + (\mathbf{A} - 2\mathbf{I})\,\frac{\sinh(5t)}{5}\right] \\ &= e^{2t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cosh(5t) + \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}\frac{\sinh(5t)}{5}\right] \\ &= e^{2t}\begin{pmatrix} \cosh(5t) + \frac{4}{5}\sinh(5t) & \frac{3}{5}\sinh(5t) \\ \frac{3}{5}\sinh(5t) & \cosh(5t) - \frac{4}{5}\sinh(5t) \end{pmatrix}. \end{aligned}$$

(3) [4] Let $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}$. Compute $e^{t\mathbf{A}}$.

Solution. The characteristic polynomial is $p(z) = z^2 - 6z + 9 = (z-3)^2$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^{3t}\big[\mathbf{I} + (\mathbf{A} - 3\mathbf{I})\,t\big] \\ &= e^{3t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} t\right] \\ &= e^{3t}\begin{pmatrix} 1 + 2t & -t \\ 4t & 1 - 2t \end{pmatrix}. \end{aligned}$$