Quiz 10 Solutions, Math 246, Professor David Levermore
Tuesday, 28 April 2009

(1) [3] $\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix}$ has eigenvalues -2 and 6. Find an eigenvector for each eigenvalue.

Solution: One has $\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 5 & 5 \\ 3 & 3 \end{pmatrix}$ and $\mathbf{A} - 6\mathbf{I} = \begin{pmatrix} -3 & 5 \\ 3 & -5 \end{pmatrix}$.
The eigenvectors $\mathbf{v}_1$ associated with the eigenvalue -2 satisfy $(\mathbf{A} + 2\mathbf{I})\mathbf{v}_1 = 0$. You can either solve this system or simply read-off from a nonzero column of $\mathbf{A} - 6\mathbf{I}$ that these have the form

$$\mathbf{v}_1 = \alpha_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{for some } \alpha_1 \neq 0 \,.$$

The eigenvectors $\mathbf{v}_2$ associated with the eigenvalue 6 satisfy $(\mathbf{A} - 6\mathbf{I})\mathbf{v}_2 = 0$. You can either solve this system or simply read-off from a nonzero column of $\mathbf{A} + 2\mathbf{I}$ that these have the form

$$\mathbf{v}_2 = \alpha_2 \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad \text{for some } \alpha_2 \neq 0 \,.$$

(2) [3] A real 2×2 matrix $\mathbf{A}$ has the eigenpair $\left(-1 + i2 \,, \begin{pmatrix} 1 \\ -i \end{pmatrix} \right)$. Diagonalize $\mathbf{A}$.

Solution: Because $\mathbf{A}$ is real, a second eigenpair will be the complex conjugate of the given eigenpair. If you use the eigenpairs

$$\left(-1 + i2 \,, \begin{pmatrix} 1 \\ -i \end{pmatrix} \right) , \qquad \left(-1 - i2 \,, \begin{pmatrix} 1 \\ i \end{pmatrix} \right) ,$$

then set

$$\mathbf{V} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} , \qquad \mathbf{D} = \begin{pmatrix} -1 + i2 & 0 \\ 0 & -1 - i2 \end{pmatrix} .$$

Because $\det(\mathbf{V}) = 1 \cdot i - (-i) \cdot 1 = 2i$, you obtain the diagonalization

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} -1 + i2 & 0 \\ 0 & -1 - i2 \end{pmatrix} \frac{1}{2i} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} .$$

(3) [4] A real 2×2 matrix $\mathbf{A}$ has the eigenpairs $\left(-1 \,, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right)$ and $\left(-2 \,, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right)$. Sketch a phase-plane portrait for the system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$. Indicate typical trajectories.

Solution: Because the eigenvalues of $\mathbf{A}$ are simple and negative, the phase-plane portrait is a *nodal sink*. (The origin is therefore attracting.)
The trajectories that lie on the lines $c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ move towards the origin. All other trajectories will move towards the origin along curves that become tangent to the line $c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ as $t \to \infty$, and become more parallel to the line $c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ as $t \to -\infty$.