Quiz 11 Solutions, Math 246, Professor David Levermore
Tuesday, 12 May 2009

(1) [5] Consider the first-order planar system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1-y)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y(x-1)\,.$$

Its stationary points are $(0,0)$ and $(1,1)$. The coefficient matrix $\mathbf{A}$ of its linearization at these points is respectively

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \qquad \mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,.$$

The first of these matrices has the real eigenpairs

$$\left(1\,, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\,, \qquad \left(-1\,, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\,,$$

while the second has the conjugate pair of eigenvalues $\pm i$. Sketch a plausible global phase-plane portrait of this system. Carefully mark all sketched orbits with arrows.

Solution: The stationary point $(0,0)$ is a *saddle*, and is therefore *unstable*, because the eigenvalues of its $\mathbf{A}$ are real, nonzero, and have opposite sign. The x-axis ($y=0$) and y-axis ($x=0$) are *invariant*. Orbits move away from $(0,0)$ along the x-axis while orbits move towards $(0,0)$ along the y-axis.

The stationary point $(1,1)$ is a *counterclockwise center*, and is therefore *stable*, because $a_{21} = 1 > 0$, and because the eigenvalues of its $\mathbf{A}$ are $\pm i$ while the system is conservative in the first quadrant — a fact seen because its integrals satisfy

$$\frac{x-1}{x}\,\mathrm{d}x - \frac{1-y}{y}\,\mathrm{d}y = 0\,,$$

which has no singularities when $x \neq 0$ and $y \neq 0$. A sketch was given in class.

(2) [5] Consider the first-order planar system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 5x + x^2 + 4y\,.$$

Its stationary points are $(0,0)$ and $(-5,0)$. Classify the type and stability of each of these stationary points. (You do not have to sketch anything.)

Solution: The matrix of partial derivatives is $\begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 5+2x & 4 \end{pmatrix}$.

- At $(0,0)$ the coefficient matrix of its linearization is $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 5 & 4 \end{pmatrix}$, which has the characteristic polynomial $p(z) = z^2 - 4z - 5 = (z+1)(z-5)$. The eigenvalues of $\mathbf{A}$ are -1 and 5. Because these are real, nonzero, and have opposite sign, the stationary point $(0,0)$ is a *saddle* and is therefore *unstable.*
- At $(-5,0)$ the coefficient matrix of its linearization is $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -5 & 4 \end{pmatrix}$, which has the characteristic polynomial $p(z) = z^2 - 4z + 5 = (z-2)^2 + 1$. The eigenvalues of $\mathbf{A}$ are the conjugate pair $2 \pm i$. Because these have postive real part, and because $a_{21} = -5 < 0$, the stationary point $(-5,0)$ is a *clockwise spiral source* and is therefore *repelling.*