

**Sample Problems for First In-Class Exam**  
**Math 246, Fall 2009, Professor David Levermore**

- (1) (a) Give the integral being evaluated by the MATLAB command

$$\text{int('x/(1+x^4)', 'x', 0, \text{inf}) .}$$

- (b) Sketch the graph that you expect would be produced by the following MATLAB commands.

```
[x, y] = meshgrid(-5:0.5:5, -5:0.2:5)
contour(x, y, x.^2 + y.^2, [1, 9, 25])
axis square
```

- (2) Find the explicit solution for each of the following initial-value problems and identify its interval of existence (definition).

(a)  $\frac{dz}{dt} = \frac{\cos(t) - z}{1 + t}, \quad z(0) = 2.$

(b)  $\frac{du}{dz} = e^u + 1, \quad u(0) = 0.$

- (3) Consider the differential equation

$$\frac{dy}{dt} = 4y^2 - y^4 .$$

- (a) Find all of its stationary (equilibrium) solutions and classify each as being either stable, unstable, or semistable.
- (b) If  $y(0) = 1$ , how does the solution  $y(t)$  behave as  $t \rightarrow \infty$ ?
- (c) If  $y(0) = -1$ , how does the solution  $y(t)$  behave as  $t \rightarrow \infty$ ?
- (d) Sketch a graph of  $y$  versus  $t$  showing the direction field and several solution curves. The graph should show all the stationary solutions as well as solution curves above and below each of them. Every value of  $y$  should lie on at least one sketched solution curve.

- (4) A tank initially contains 100 liters of pure water. Beginning at time  $t = 0$  brine (salt water) with a salt concentration of 2 grams per liter (g/l) flows into the tank at a constant rate of 3 liters per minute (l/min) and the well-stirred mixture flows out of the tank at the same rate. Let  $S(t)$  denote the mass (g) of salt in the tank at time  $t \geq 0$ .

- (a) Write down an initial-value problem that governs  $S(t)$ .
- (b) Is  $S(t)$  an increasing or decreasing function of  $t$ ? (Give your reasoning.)
- (c) What is the behavior of  $S(t)$  as  $t \rightarrow \infty$ ? (Give your reasoning.)
- (d) Derive an explicit formula for  $S(t)$ .

- (5) Suppose you are using the Heun-midpoint method to numerically approximate the solution of an initial-value problem over the time interval  $[0, 5]$ . By what factor would you expect the error to decrease when you increase the number of time steps taken from 500 to 2000.
- (6) Give an implicit general solution to each of the following differential equations.
- (a)  $\left(\frac{y}{x} + 3x\right) dx + (\log(x) - y) dy = 0.$
- (b)  $(x^2 + y^3 + 2x) dx + 3y^2 dy = 0.$
- (7) A 2 kilogram (kg) mass initially at rest is dropped in a medium that offers a resistance of  $v^2/40$  newtons ( $= \text{kg m/sec}^2$ ) where  $v$  is the downward velocity (m/sec) of the mass. The gravitational acceleration is  $9.8 \text{ m/sec}^2$ .
- (a) What is the terminal velocity of the mass?
- (b) Write down an initial-value problem that governs  $v$  as a function of time. (You do not have to solve it!)
- (8) Consider the following MATLAB function M-file.

```
function [t,y] = solveit(ti, yi, tf, n)

h = (tf - ti)/n;
t = zeros(n + 1, 1);
y = zeros(n + 1, 1);
t(1) = ti;
y(1) = yi;
for i = 1:n
z = t(i)^4 + y(i)^2;
t(i + 1) = t(i) + h;
y(i + 1) = y(i) + (h/2)*(z + t(i + 1)^4 + (y(i) + h*z)^2);
end
```

- (a) What is the initial-value problem being approximated numerically?
- (b) What is the numerical method being used?
- (c) What are the output values of  $t(2)$  and  $y(2)$  that you would expect for input values of  $t_i = 1$ ,  $y_i = 1$ ,  $t_f = 5$ ,  $n = 20$ ?