Quiz 4 Solutions, Math 246, Professor David Levermore
Tuesday, 6 October 2009

(1) [3] What is the interval of definition for the solution to the initial-value problem

$$(t^2-4)\frac{\mathrm{d}^4w}{\mathrm{d}t^4}+4tw=\frac{1}{\sin(t)}\,,\qquad w(3)=w'(3)=w''(3)=w'''(3)=5\,.$$

Solution: The linear normal form is $\dfrac{\mathrm{d}^4w}{\mathrm{d}t^4}+\dfrac{4t}{(t+2)(t-2)}\,w=\dfrac{1}{(t+2)(t-2)\sin(t)}$. The coefficient is defined and continuous everywhere except $t=-2$ and $t=2$. The forcing is defined and continuous everywhere except $t=-2$, $t=2$, and $t=k\pi$ for some integer k. The initial time is $t=3$. The interval of definition is therefore $(2,\pi)$.

(2) [3] Compute the Wronskian $W[Y_1,Y_2](t)$ of the functions $Y_1(t)=t+1$ and $Y_2(t)=e^t$. (Evaluate the determinant and simplify.)

Solution: Because $Y_1'(t)=1$ and $Y_2'(t)=e^t$, the Wronskian is

$$\begin{aligned}W[Y_1,Y_2](t)&=\det\begin{pmatrix}Y_1(t)&Y_2(t)\\Y_1'(t)&Y_2'(t)\end{pmatrix}=\det\begin{pmatrix}t+1&e^t\\1&e^t\end{pmatrix}\\&=(t+1)e^t-1e^t=te^t\,.\end{aligned}$$

(3) [3] Given that $\cos(4t)$ and $\sin(4t)$ are linearly independent solutions of $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}+16y=0$, find the solution $Y(t)$ that satisfies the initial conditions $Y(0)=3$, $Y'(0)=8$.

Solution: Let $Y(t)=c_1\cos(4t)+c_2\sin(4t)$. Then $Y'(t)=-4c_1\sin(4t)+4c_2\cos(4t)$. To satisfy the initial conditions one needs

$$3=Y(0)=c_1\,,\qquad\qquad 8=Y'(0)=4c_2\,.$$

It follows that $c_1=3$ and $c_2=2$. The solution of the initial-value problem is therefore

$$Y(t)=3\cos(4t)+2\sin(4t)\,.$$

(4) [1] Suppose that $Y_1(t)$, $Y_2(t)$, and $Y_3(t)$ are solutions of the differential equation

$$y'''+a(t)y'=0\,,$$

where $a(t)$ is continuous over $(-4,7)$. Suppose you know that $W[Y_1,Y_2,Y_3](0)=2$. What is $W[Y_1,Y_2,Y_3](5)$?

Solution: Because the equation is third-order while the coefficient of y'' is zero, Abel's Theorem implies that $W[Y_1,Y_2,Y_3](t)$ is constant over $(-4,7)$. We can thereby conclude that $W[Y_1,Y_2,Y_3](5)=W[Y_1,Y_2,Y_3](0)=2$.