Quiz 7 Solutions, Math 246, Professor David Levermore
Tuesday, 27 October 2009

(1) [5] The functions t and t^2 are solutions of the homogeneous equation

$$t^2 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 2t \frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 0 \qquad \text{over } t > 0\,.$$

(You do not have to check that this is true!) Find an explicit particular solution of

$$t^2 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 2t \frac{\mathrm{d}y}{\mathrm{d}t} + 2y = \frac{t^3}{1+t^2}\,, \qquad \text{over } t > 0\,.$$

Solution. Because this problem has variable coefficients, you must use either the general Green function method or the variation of parameters method to solve it. To apply either method you must first bring the equation into its normal form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{2}{t}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{2}{t^2}\,y = \frac{t}{1+t^2}\,, \qquad \text{over } t > 0\,.$$

General Green Function. The Green function $G(t,s)$ is given by

$$G(t,s) = \frac{\det\begin{pmatrix} s & s^2 \\ t & t^2 \end{pmatrix}}{\det\begin{pmatrix} s & s^2 \\ 1 & 2s \end{pmatrix}} = \frac{st^2 - ts^2}{2s^2 - s^2} = \frac{t^2 - ts}{s}\,.$$

The Green function formula then yields the particular solution

$$\begin{aligned} y(t) &= \int_0^t G(t,s)\,\frac{s}{1+s^2}\,\mathrm{d}s = \int_0^t \frac{t^2 - ts}{1+s^2}\,\mathrm{d}s = t^2 \int_0^t \frac{1}{1+s^2}\,\mathrm{d}s - t\int_0^t \frac{s}{1+s^2}\,\mathrm{d}s \\ &= t^2 \tan^{-1}(t) - \tfrac{1}{2} t \log(1+t^2)\,. \end{aligned}$$

Variation of Parameters. A general solution of the associated homogeneous problem is

$$y_H(t) = c_1 t + c_2 t^2\,.$$

Seek a solution in the form

$$y = u_1(t) t + u_2(t) t^2\,,$$

where $u_1'(t)$ and $u_2'(t)$ satisfy

$$\begin{aligned} u_1'(t) t + u_2'(t) t^2 &= 0\,, \\ u_1'(t) 1 + u_2'(t) 2t &= \frac{t}{1+t^2}\,. \end{aligned}$$

Solve this system to obtain

$$u_1'(t) = -\frac{t}{1+t^2}\,, \qquad u_2'(t) = \frac{1}{1+t^2}\,.$$

Integrate these equations to find

$$u_1(t) = c_1 - \tfrac{1}{2}\log(1+t^2)\,, \qquad u_2(t) = c_2 + \tan^{-1}(t)\,.$$

A particular solution is therefore

$$y(t) = -\tfrac{1}{2} t \log(1+t^2) + t^2 \tan^{-1}(t)\,.$$

(2) [5] When a 2 kilogram mass is hung vertically from a spring, at rest it stretches the spring .098 m. (Assume that the spring force is proportional to displacement and that gravitational acceleration is $g = 9.8$ m/sec^2.) At $t = 0$ the mass is displaced .4 m below its equilibrium position and is released with an upward velocity of .3 m/sec. There are no other forces. Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

Solution. Let $h(t)$ be the displacement of the mass from its equilibrium (rest) position at time t in meters, with upward displacements being positive. The governing initial-value problem then has the form

$$m\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + kh = 0\,, \qquad h(0) = -.4\,, \quad h'(0) = .3\,,$$

where m is the mass and k is the spring constant. The problem states that $m = 2$ kilograms. The spring constant is obtained by balancing the weight of the mass ($mg = 2 \cdot 9.8$ Newtons) with the force applied by the spring when it is stetched .098 m. This gives $k\,.098 = 2 \cdot 9.8$, or

$$k = \frac{2 \cdot 9.8}{.098} = 200 \quad \text{Newt/m}\,.$$

The governing initial-value problem is therefore

$$2\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + 200h = 0\,, \qquad h(0) = -.4\,, \quad h'(0) = .3\,.$$