Quiz 8 Solutions, Math 246, Professor David Levermore
Tuesday, 10 November 2009

Short Table: $\mathcal{L}[\cos(bt)](s) = \dfrac{s}{s^2+b^2}$ for $s > 0$, $\mathcal{L}[e^{at}](s) = \dfrac{1}{s-a}$ for $s > a$.

(1) [3] Use the definition of the Laplace transform to compute $\mathcal{L}[f](s)$ for $f(t) = u(t-4)$, where u is the unit step function.

Solution: By the definitions of the Laplace transform and the unit step function

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} u(t-4)\,\mathrm{d}t = \lim_{T\to\infty} \int_4^T e^{-st}\,\mathrm{d}t .$$

The integral diverges for $s \le 0$, while for $s > 0$ one obtains

$$\int_4^T e^{-st}\,\mathrm{d}t = -\frac{e^{-st}}{s}\bigg|_4^T = \frac{e^{-s4}}{s} - \frac{e^{-sT}}{s} ,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\left[\frac{e^{-s4}}{s} - \frac{e^{-sT}}{s}\right] = \frac{e^{-4s}}{s} \quad \text{for } s > 0 .$$

(2) [4] Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 4y = \cos(5t) , \qquad y(0) = y'(0) = 0 . \qquad \text{DO NOT solve for } y(t), \text{ just } Y(s)!$$

Solution: The Laplace transform of the initial-value problem and item 1 in the table at the top of the page with $b = 5$ gives

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y](s) = \mathcal{L}[\cos(5t)](s) = \frac{s}{s^2+5^2} = \frac{s}{s^2+25} ,$$

where

$$\mathcal{L}[y](s) = Y(s) ,$$
$$\mathcal{L}[y''](s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) .$$

Hence,

$$(s^2+4)Y(s) = \frac{s}{s^2+25} , \qquad \Longrightarrow \qquad Y(s) = \frac{s}{(s^2+4)(s^2+25)} .$$

(3) [3] Find the inverse Laplace transform $y(t)$ of the function $Y(s) = \dfrac{s-1}{s^2-2s-15}$.

Solution: By partial fractions

$$Y(s) = \frac{s-1}{s^2-2s-15} = \frac{s-1}{(s-5)(s+3)} = \frac{\frac{1}{2}}{s-5} + \frac{\frac{1}{2}}{s+3} .$$

Item 2 in the table at the top of the page with $a = 5$ and with $a = -3$ then gives

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y](t) &= \mathcal{L}^{-1}\left[\frac{\frac{1}{2}}{s-5} + \frac{\frac{1}{2}}{s+3}\right](t) \\ &= \tfrac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s-5}\right](t) + \tfrac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+3}\right](t) = \tfrac{1}{2}e^{5t} + \tfrac{1}{2}e^{-3t} . \end{aligned}$$