Quiz 9 Solutions, Math 246, Professor David Levermore
Tuesday, 17 November 2009

(1) [4] Let $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 5 \\ -5 & 2 \end{pmatrix}$.

(a) Compute $2\mathbf{A} - \mathbf{B}$.

Solution:

$$2\mathbf{A} - \mathbf{B} = \begin{pmatrix} 8 & 6 \\ 6 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 5 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ 11 & 2 \end{pmatrix} .$$

(b) Compute $\mathbf{AB}$.

Solution:

$$\mathbf{AB} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 4-15 & 20+6 \\ 3-10 & 15+4 \end{pmatrix} = \begin{pmatrix} -11 & 26 \\ -7 & 19 \end{pmatrix} .$$

(2) [2] Transform the equation $\dfrac{\mathrm{d}^3 v}{\mathrm{d}t^3} + t^2 \dfrac{\mathrm{d}v}{\mathrm{d}t} - t^3 v = e^t$ into a first-order system of ordinary differential equations.

Solution: Because the equation is third order, the first order system must have dimension three. The simplest such first order system is

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ t^3 x_1 - t^2 x_2 + e^t \end{pmatrix} , \qquad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} v \\ v' \\ v'' \end{pmatrix} .$$

(3) [4] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^3 + 1 \\ t \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 1 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

Solution.

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{pmatrix} t^3 + 1 & t^2 \\ t & 1 \end{pmatrix} = (t^3 + 1) - t^3 = 1 .$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$.

Solution. Let $\mathbf{X}(t) = \begin{pmatrix} t^3 + 1 & t^2 \\ t & 1 \end{pmatrix}$. Because $\dfrac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}(t) = \mathbf{A}(t)\mathbf{X}(t)$, one has

$$\begin{aligned} \mathbf{A}(t) &= \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}(t)\, \mathbf{X}(t)^{-1} = \begin{pmatrix} 3t^2 & 2t \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t^3 + 1 & t^2 \\ t & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 3t^2 & 2t \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -t \\ -t^2 & t^3 + 1 \end{pmatrix} = \begin{pmatrix} t^2 & 2t - t^4 \\ 1 & -t^2 \end{pmatrix} . \end{aligned}$$