Quiz 10 Solutions, Math 246, Professor David Levermore
Tuesday, 24 November 2009

(1) [4] Let $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$. Compute $e^{t\mathbf{A}}$.

Solution. The characteristic polynomial is $p(z) = z^2 - 4z + 4 = (z-2)^2$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^{2t}\big[\mathbf{I} + (\mathbf{A} - 2\mathbf{I})\, t\big] \\ &= e^{2t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} t\right] \\ &= e^{2t}\begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix} . \end{aligned}$$

(2) [3] $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ has eigenvalues -2 and 2. Find an eigenvector for each eigenvalue.

Solution: One has $\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$.
The eigenvectors $\mathbf{v}_1$ associated with the eigenvalue -2 satisfy $(\mathbf{A} + 2\mathbf{I})\mathbf{v}_1 = 0$. You can either solve this system or simply read-off from a nonzero column of $\mathbf{A} - 2\mathbf{I}$ that these have the form

$$\mathbf{v}_1 = \alpha_1 \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad \text{for some } \alpha_1 \neq 0 \,.$$

The eigenvectors $\mathbf{v}_2$ associated with the eigenvalue 2 satisfy $(\mathbf{A} - 2\mathbf{I})\mathbf{v}_2 = 0$. You can either solve this system or simply read-off from a nonzero column of $\mathbf{A} + 2\mathbf{I}$ that these have the form

$$\mathbf{v}_2 = \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for some } \alpha_2 \neq 0 \,.$$

(3) [3] A real 2×2 matrix $\mathbf{A}$ has the eigenpair $\left(-1 + i3 \,,\, \begin{pmatrix} 2 \\ -i \end{pmatrix}\right)$. Diagonalize $\mathbf{A}$.

Solution: Because $\mathbf{A}$ is real, a second eigenpair will be the complex conjugate of the given eigenpair. If you use the eigenpairs

$$\left(-1 + i3 \,,\, \begin{pmatrix} 2 \\ -i \end{pmatrix}\right) , \qquad \left(-1 - i3 \,,\, \begin{pmatrix} 2 \\ i \end{pmatrix}\right) ,$$

then set

$$\mathbf{V} = \begin{pmatrix} 2 & 2 \\ -i & i \end{pmatrix} , \qquad \mathbf{D} = \begin{pmatrix} -1 + i3 & 0 \\ 0 & -1 - i3 \end{pmatrix} .$$

Because $\det(\mathbf{V}) = 2 \cdot i - (-i) \cdot 2 = 4i$, you obtain the diagonalization

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \begin{pmatrix} 2 & 2 \\ -i & i \end{pmatrix} \begin{pmatrix} -1 + i3 & 0 \\ 0 & -1 - i3 \end{pmatrix} \frac{1}{4i} \begin{pmatrix} i & -2 \\ i & 2 \end{pmatrix} .$$