First In-Class Exam Solutions: Math 410
Section 0401, Professor Levermore
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1. [10] Let X be a field. Use the field axioms to show that if $x \in X$ then $x0 = 0$.

   Remark: The main point to keep in mind when doing problems like this is *to justify every step in your solution either by one or more of the axioms or by a previous step.*

   Solution. Let $x \in X$. The additive identity axiom implies $0 = 0 + 0$. The distributive axiom then gives the equality

$$x0 = x(0 + 0) = x0 + x0\,.$$

   Then

$$\begin{aligned} 0 &= x0 + \big(-(x0)\big) && \text{(add. inv. axiom)} \\ &= (x0 + x0) + \big(-(x0)\big) && \text{(above equality)} \\ &= x0 + \big(x0 + \big(-(x0)\big)\big) && \text{(add. assoc. axiom)} \\ &= x0 + 0 && \text{(add. inv. axiom)} \\ &= x0 && \text{(add. indent. axiom)}\,. \end{aligned}$$

   □

2. [10] Suppose that $a \in \mathbb{R}$ has the property that $a < 1/k$ for every $k \in \mathbb{Z}_+$. Prove $a \leq 0$.

   Solution. Suppose $a \leq 0$ does not hold. Then by trichotomy $a > 0$. By the Archimedean Property there exists $n \in \mathbb{Z}_+$ such that $1 < na$. Then $1/n < a$, which contradicts the property that $a < 1/k$ for every $k \in \mathbb{Z}_+$. Therefore $a \leq 0$ holds. □

   Remark. An alternative solution that uses more advanced machinery (and therefore is not as good) is the following. Because constant sequences converge while $1/k \to 0$ as $k \to \infty$, and because of the way limits preserve inequalities, one has

$$a = \lim_{k\to\infty} a \leq \lim_{k\to\infty} 1/k = 0\,. \quad \square$$

   The Archimedean Property lies behind the fact that $1/k \to 0$ as $k \to \infty$ in this alternative solution. □

3. [10] Write down a counterexample to each of the following assertions.
   (a) A sequence $\{a_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ is convergent if the sequence $\{a_k^2\}_{k\in\mathbb{N}}$ is convergent.
   (b) A countable union of closed subsets of $\mathbb{R}$ is closed.
   (c) Every convergent series in $\mathbb{R}$ is absolutely convergent.

   Solution (a). Let $a_k = (-1)^k$ for every $k \in \mathbb{N}$. Then the sequence $\{a_k^2\}_{k\in\mathbb{N}}$ converges to 1 (because $a_k^2 = (-1)^{2k} = 1$), while the sequence $\{a_k\}_{k\in\mathbb{N}}$ diverges. □

   Solution (b). Let $I_k = [2^{-k}, 2]$ for every $k \in \mathbb{N}$. Then each interval I_k is closed while

$$\bigcup_{k\in\mathbb{N}} I_k = (0, 2] \quad \text{is not closed}\,.$$

   □

Solution (c). The series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \quad \text{is convergent}\,,$$

by the Alternating Series Test, but is not absolutely convergent because the harmonic series,

$$\sum_{k=1}^{\infty} \frac{1}{k}\,, \quad \text{is divergent}\,. \quad \square$$

4. [10] Consider the real sequence $\{b_k\}_{k\in\mathbb{N}}$ given by

$$b_k = (-1)^k \frac{2k+4}{k+1} \qquad \text{for every } k \in \mathbb{N} = \{0, 1, 2, \dots\}\,.$$

(a) Write down the first three terms of the subsequence $\{b_{2k}\}_{k\in\mathbb{N}}$.
(b) Write down the first three terms of the subsequence $\{b_{2^k}\}_{k\in\mathbb{N}}$.
(c) Write down $\liminf_{k\to\infty} b_k$ and $\limsup_{k\to\infty} b_k$. (No proof is needed here.)

Solution. You are given that $\mathbb{N} = \{0, 1, 2, \cdots\}$, as was done in class and in the notes (but in not the book). Then (a) the first three terms of the subsequence $\{b_{2k}\}_{k\in\mathbb{N}}$ are

$$b_0 = 4\,, \qquad b_2 = \tfrac{8}{3}\,, \qquad b_4 = \tfrac{12}{5}\,,$$

while (b) the first three terms of the subsequence $\{b_{2^k}\}_{k\in\mathbb{N}}$ are

$$b_1 = -3\,, \qquad b_2 = \tfrac{8}{3}\,, \qquad b_4 = \tfrac{12}{5}\,.$$

Because $b_{2k+1} < -2$ while $b_{2k} > 2$, and because

$$\lim_{k\to\infty} b_{2k+1} = -\lim_{k\to\infty} \frac{4k+6}{2k+2} = -2\,,$$

while

$$\lim_{k\to\infty} b_{2k} = \lim_{k\to\infty} \frac{4k+4}{2k+1} = 2\,,$$

(c) one has that

$$\liminf_{k\to\infty} b_k = -2\,, \qquad \limsup_{k\to\infty} b_k = 2\,.$$

$\square$

5. [10] Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be bounded sequences in $\mathbb{R}$.
(a) Prove that

$$\liminf_{k\to\infty} a_k + \liminf_{k\to\infty} b_k \le \liminf_{k\to\infty} (a_k + b_k)\,.$$

(b) Write down an example for which equality does not hold above.

Solution (a): Let $c_k = a_k + b_k$ for every $k \in \mathbb{N}$. By the definition of lim inf we have

$$\liminf_{k\to\infty} a_k = \lim_{k\to\infty} \underline{a}_k\,, \qquad \liminf_{k\to\infty} b_k = \lim_{k\to\infty} \underline{b}_k\,, \qquad \liminf_{k\to\infty} c_k = \lim_{k\to\infty} \underline{c}_k\,,$$

where for every $k \in \mathbb{N}$ we define

$$\underline{a}_k = \inf\{a_l \,:\, l \ge k\}\,, \qquad \underline{b}_k = \inf\{b_l \,:\, l \ge k\}\,, \qquad \underline{c}_k = \inf\{c_l \,:\, l \ge k\}\,.$$

Because the sequences $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$, and $\{c_k\}_{k\in\mathbb{N}}$ are bounded below, for every $k \in \mathbb{N}$ we have

$$-\infty < \underline{a}_k\,, \qquad -\infty < \underline{b}_k\,, \qquad -\infty < \underline{c}_k\,.$$

Therefore $\{\underline{a}_k\}_{k\in\mathbb{N}}$, $\{\underline{b}_k\}_{k\in\mathbb{N}}$, and $\{\underline{b}_k\}_{k\in\mathbb{N}}$ are nondecreasing sequences in $\mathbb{R}$. Moreover, they are bounbed above because $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$, and $\{c_k\}_{k\in\mathbb{N}}$ are bounded above. Hence, they converge by the Montonic Sequence Theorem:

$$\liminf_{k\to\infty} a_k = \lim_{k\to\infty} \underline{a}_k\,, \qquad \liminf_{k\to\infty} b_k = \lim_{k\to\infty} \underline{b}_k\,, \qquad \liminf_{k\to\infty} c_k = \lim_{k\to\infty} \underline{c}_k\,.$$

Because for every $k \in \mathbb{N}$ we have

$$\underline{a}_k + \underline{b}_k \leq a_l + b_l = c_l \quad \text{for every } l \geq k\,,$$

the sequences $\{\underline{a}_k\}_{k\in\mathbb{N}}$, $\{\underline{b}_k\}_{k\in\mathbb{N}}$, and $\{\underline{b}_k\}_{k\in\mathbb{N}}$ thereby also satisfy the inequality

$$\underline{a}_k + \underline{b}_k \leq \inf\{c_l \,:\, l \geq k\} = \underline{c}_k\,.$$

Then by the properties of limits

$$\begin{aligned}\liminf_{k\to\infty} a_k + \liminf_{k\to\infty} b_k &= \lim_{k\to\infty} \underline{a}_k + \lim_{k\to\infty} \underline{b}_k = \lim_{k\to\infty} (\underline{a}_k + \underline{b}_k) \\ &\leq \lim_{k\to\infty} \underline{c}_k = \liminf_{k\to\infty} c_k = \liminf_{k\to\infty} (a_k + b_k)\,.\end{aligned}$$

□

Solution (b): Let $a_k = (-1)^k$ and $b_k = (-1)^{k+1}$ for every $k \in \mathbb{N}$. Clearly

$$\liminf_{k\to\infty} a_k = \lim_{k\to\infty} a_{2k+1} = -1\,, \qquad \liminf_{k\to\infty} b_k = \lim_{k\to\infty} b_{2k} = -1\,,$$

while (because $a_k + b_k = 0$ for every $k \in \mathbb{N}$)

$$\liminf_{k\to\infty} (a_k + b_k) = \lim_{k\to\infty} (a_k + b_k) = 0\,.$$

Therefore

$$\liminf_{k\to\infty} a_k + \liminf_{k\to\infty} b_k = -2 < 0 = \liminf_{k\to\infty} (a_k + b_k)\,.$$

□

6. [10] Let A and B be subsets of $\mathbb{R}$.
   (a) Prove that $(A \cap B)^c \subset A^c \cap B^c$.
   (b) Write down an example for which equality does not hold above.

Solution (a): Let $x \in (A \cap B)^c$. By the definition of closure, there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ contained in $A \cap B$ such that $x_k \to x$ as $k \to \infty$. But the sequence $\{x_k\}_{k\in\mathbb{N}}$ is therefore contained in both A and B while $x_k \to x$ as $k \to \infty$. By the definition of closure, it follows that $x \in A^c$ and $x \in B^c$, whereby $x \in A^c \cap B^c$. □

Solution (b): A simple example is $A = (0,1)$ and $B = (1,2)$. Then $(A \cap B)^c = \emptyset^c = \emptyset$ (because $A \cap B = \emptyset$), while $A^c \cap B^c = [0,1] \cap [1,2] = \{1\}$ (because $A^c = [0,1]$ and $B^c = [1,2]$). Hence, $\emptyset = (A \cap B)^c \neq A^c \cap B^c = \{1\}$. □

Remark. A more dramatic example is $A = \mathbb{Q}$ and $B = \{\sqrt{2} + q \,:\, q \in \mathbb{Q}\}$. Then $(A \cap B)^c = \emptyset^c = \emptyset$ (because $A \cap B = \emptyset$), while $A^c \cap B^c = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$ (because $A^c = \mathbb{R}$ and $B^c = \mathbb{R}$). Hence, $\emptyset = (A \cap B)^c \neq A^c \cap B^c = \mathbb{R}$. □

7. [20] Determine all $a \in \mathbb{R}$ for which the following formal infinite series converge. Give your reasoning.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{a^n}{n}$

(b) $\displaystyle\sum_{k=0}^{\infty} \left(\frac{k^2+1}{k^6+1}\right)^a$

Solution (a). The series converges for $a \in [-1, 1)$ and diverges otherwise.

The cases $|a| < 1$ and $|a| > 1$ are best handled by the Ratio Test. Let $b_n = a^n/n$. Because

$$\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = \lim_{n\to\infty} \frac{n}{n+1}|a| = |a|\,,$$

the Ratio Test shows that this series converges absolutely for $|a| < 1$ and diverges for $|a| > 1$. The Root Test would lead to the same conclusions.

The case $a = -1$ is best handled by the Alternating Series Test. Indeed, because the sequence

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \quad \text{is decreasing and positive}\,, \qquad \text{while} \qquad \lim_{n\to\infty} \frac{1}{n} = 0\,,$$

the Alternating Series Test shows that

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \quad \text{converges}\,.$$

The case $a = 1$ reduces to the harmonic series, which diverges. Had you forgotten this fact, you could recover it with either the Cauchy 2^k Test or the Integral Test. □

Solution (b). The series converges for $a \in (\frac{1}{4}, \infty)$ and diverges otherwise. Because

$$\frac{k^2+1}{k^6+1} \sim \frac{1}{k^4} \quad \text{as } k \to \infty\,,$$

one sees that the original series should be compared with the p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^{4a}}\,.$$

This is best handled by Limit Comparison Test. Indeed, because for every $a \in \mathbb{R}$ one has

$$\lim_{k\to\infty} \frac{\left(\dfrac{k^2+1}{k^6+1}\right)^a}{\dfrac{1}{k^{4a}}} = \lim_{k\to\infty} \left(\frac{k^6+k^4}{k^6+1}\right)^a = 1\,,$$

the Limit Comparison Test then implies that

$$\sum_{k=1}^{\infty} \left(\frac{k^2+1}{k^6+1}\right)^a \quad \text{converges} \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} \frac{1}{k^{4a}} \quad \text{converges}\,.$$

Because the $p = 4a$ for the p-series, it converges for $a \in (\frac{1}{4}, \infty)$ and diverges otherwise. The same is therefore true for the original series. □

8. [10] Let $\{a_k\}_{k\in\mathbb{N}}$ be a real sequence and $\{a_{n_k}\}$ be any subsequence of it. Show that

$$\sum_{k=0}^{\infty} a_k \quad \text{converges absolutely} \qquad \Longrightarrow \qquad \sum_{k=0}^{\infty} a_{n_k} \quad \text{converges absolutely}\,.$$

Solution. For every $m, n \in \mathbb{N}$ define the sequences $\{p_m\}$ and $\{q_n\}$ of partial sums

$$p_m = \sum_{k=0}^{m} |a_{n_k}|\,, \qquad q_n = \sum_{k=0}^{n} |a_k|\,.$$

It is clear from these definitions that these sequences are nondecreasing and satisfy

$$p_m = \sum_{k=0}^{m} |a_{n_k}| \le \sum_{k=0}^{n_m} |a_k| = q_{n_m} \quad \text{for every } m \in \mathbb{N}\,.$$

It then follows from the definition of absolute convergence, the Monotonic Sequence Theorem, and the above inequality that

$$\begin{aligned}
\sum_{k=0}^{\infty} a_k \quad \text{converges absolutely} \qquad &\Longleftrightarrow \qquad \{q_n\} \text{ converges} \\
&\Longleftrightarrow \qquad \{q_n\} \text{ is bounded above} \\
&\Longrightarrow \qquad \{p_m\} \text{ is bounded above} \\
&\Longleftrightarrow \qquad \{p_m\} \text{ converges} \\
&\Longleftrightarrow \qquad \sum_{k=0}^{\infty} a_{n_k} \quad \text{converges absolutely}\,.
\end{aligned}$$

□

9. [10] Let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let A be a subset of $\mathbb{R}$. Write the negations of the following assertions.
   (a) "For some $\epsilon > 0$ one has $|b_j - 3| \ge \epsilon$ frequently as $j \to \infty$."
   (b) "Every sequence in A has a subsequence that converges to a limit in A."

Solution (a). "For every $\epsilon > 0$ one has $|b_j - 3| < \epsilon$ eventually as $j \to \infty$." □

Solution (b). "There is a sequence in A such that no subsequence of it converges to a limit in A." □

Or better

"There is a sequence in A such that every subsequence of it either diverges or converges to a limit outside A." □