Final Exam Solutions: MATH 410
Thursday, 17 December 2009
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Remark. This was the hardest final exam I have ever given to a Math 410 class, mostly due to its length. It was a challenge to an exceptional class. Many in the class rose to the challenge, with about two thirds earning an A or B. Every problem was done correctly by at least one student, although no student did them all correctly. I do not plan to give a Math 410 final exam that is this hard again. — D.L.

1. [10] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Give negations of each of the following assertions.

   (a) For every $\epsilon > 0$ there exists an $n_\epsilon \in \mathbb{N}$ such that
   $$m, n > n_\epsilon \implies |x_m - x_n| < \epsilon\,.$$
   Solution. There exists an $\epsilon > 0$ such that for every $N \in \mathbb{N}$ there exists $m, n \in \mathbb{N}$ such that
   $$m, n > N \quad \text{and} \quad |x_m - x_n| \geq \epsilon\,.$$

   (b) $\lim_{n\to\infty} x_n = \infty\,.$

   Solution. There are several acceptable answers. The shortest is
   $$\liminf_{n\to\infty} x_n < \infty\,.$$
   This could be expanded as
   $$\exists M > 0 \quad \text{such that} \quad x_n \leq M \quad \text{frequently as } n \to \infty\,,$$
   which could be expanded further as
   $$\exists M > 0 \quad \text{such that} \quad \forall m \in \mathbb{N} \quad \exists n > m \quad \text{such that} \quad x_n \leq M\,.$$
   You can also obtain the last two answers by first expressing $\lim_{n\to\infty} x_n = \infty$ either as
   $$\forall M > 0 \quad x_n > M \quad \text{eventually as } n \to \infty\,,$$
   or as
   $$\forall M > 0 \quad \exists m \in \mathbb{N} \quad \text{such that} \quad \forall n > m \qquad x_n > M\,,$$
   and then simply negating.

2. [10] Let $\{a_k\}_{k\in\mathbb{N}}$ be a nondecreasing sequence in $\mathbb{R}$. Show that it converges if it has a converging subsequence.

   Solution. Let $\{a_{n_k}\}_{k\in\mathbb{N}}$ be a converging subsequence of $\{a_k\}_{k\in\mathbb{N}}$. Because every subsequence of a nondecreasing sequence is also nondecreasing, the Monotonic Sequence Theorem states that the convergence of $\{a_{n_k}\}_{k\in\mathbb{N}}$ implies that
   $$\lim_{k\to\infty} a_{n_k} = \sup\{a_{n_k} \,:\, k \in \mathbb{N}\} < \infty\,.$$
   Because for every $k \in \mathbb{N}$ we have $k \leq n_k$, the fact $\{a_k\}_{k\in\mathbb{N}}$ is nondecreasing implies that
   $$a_k \leq a_{n_k} \leq \sup\{a_{n_k} \,:\, k \in \mathbb{N}\} < \infty\,.$$
   The nondecreasing sequence $\{a_k\}_{k\in\mathbb{N}}$ is thereby bounded above, and therefore converges by the Monotonic Sequence Theorem. □

3. [20] State whether each of the following statements is true or false. Give a proof when true and a counterexample when false.

   (a) If the interval (a,b) is bounded, $f:(a,b)\to\mathbb{R}$ is differentiable, and $f':(a,b)\to\mathbb{R}$ is bounded over (a,b) then the function f is bounded over (a,b).

   Solution. This statement is *true*. First notice that because $f:(a,b)\to\mathbb{R}$ is differentiable the Mean-Value Theorem implies that for every x, $y\in(a,b)$ there exists $p\in(a,b)$ between x and y such that

$$f(x)-f(y)=f'(p)(x-y)\,.$$

   Because $f':(a,b)\to\mathbb{R}$ is bounded this implies that for every x, $y\in(a,b)$ one has

$$|f(x)-f(y)|=M\,|x-y|\,,$$

   where $M=\sup\{|f'(x)|\,:\,x\in(a,b)\}$. In other words, f is Lipschitz continuous over (a,b). This was a theorem from the notes that you could have just cited.

   Finally, any function that is Lipschitz continuous over a bounded subset of $\mathbb{R}$ is also bounded. Indeed, pick any $c\in(a,b)$. Then for every $x\in(a,b)$ one has the bound

$$|f(x)|\le|f(c)|+|f(x)-f(c)|\le|f(c)|+M\,|x-c|\le|f(c)|+M\,(b-a)\,.$$

   More generally, any function that is uniformly continuous over a bounded subset of $\mathbb{R}$ is also bounded. ☐

   (b) If $\{f_n\}_{n=1}^\infty$ is a sequence of functions such that each $f_n:[a,b]\to\mathbb{R}$ is differentiable over $[a,b]$, and $f_n\to f$ uniformly over $[a,b]$ where $f:[a,b]\to\mathbb{R}$ is differentiable over $[a,b]$, then

$$\lim_{n\to\infty}f_n'(x)=f'(x)\quad\text{for every } x\in[a,b]\,.$$

   Solution. This statement is *false*. There are many counterexamples. A simple one is

$$f_n(x)=\tfrac{1}{n}\,e^{-nx}\to 0=f(x)\quad\text{uniformly over } [0,1] \text{ because } |f_n(x)|\le\tfrac{1}{n}\,,$$

   but

$$f_n'(0)=-e^{-nx}\Big|_{x=0}=-1\neq 0=f'(0)\,.$$

   A more dramatic counterexample is

$$f_n(x)=\frac{1}{2^n}\sin(2^nx)\to 0=f(x)\quad\text{uniformly over } \mathbb{R} \text{ because } |f_n(x)|\le\tfrac{1}{2^n}\,,$$

   but if $x=2^{-k}m\pi$ for some k, $m\in\mathbb{N}$ then

$$f_n'(x)=\cos(2^{n-k}m\pi)\to 1\neq 0=f'(x)\,.$$

   Because the set of all points having the form $2^{-k}m\pi$ for some k, $m\in\mathbb{N}$ is dense in $\mathbb{R}$, this example works when the functions f_n and f are restricted to any interval $[a,b]\subset\mathbb{R}$ with $a<b$. ☐

4. [20] Let $f : (a, b) \to \mathbb{R}$ be differentiable at a point $c \in (a, b)$ with $f'(c) > 0$. Show that there exists a $\delta > 0$ such that

$$x \in (c - \delta, c) \subset (a, b) \implies f(x) < f(c)\,,$$
$$x \in (c, c + \delta) \subset (a, b) \implies f(c) < f(x)\,,$$

Remark. It is very incorrect to assert that f is decreasing in an interval containing c. You are being asked to prove the "Transversality Lemma" from the notes.

Solution. Because f is differentiable at c, we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)\,.$$

Because $f'(c) > 0$ the ϵ-δ characterization of this limit with $\epsilon = f'(c)$ implies that there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subset (a, b)$ and

$$\begin{aligned} 0 < |x - c| < \delta \qquad &\implies \qquad \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < f'(c) \\ &\iff \qquad 0 < \frac{f(x) - f(c)}{x - c} < 2f'(c)\,. \end{aligned}$$

Hence,

$$\begin{aligned} x \in (c - \delta, c) &\implies f(x) - f(c) = \frac{f(x) - f(c)}{x - c}(x - c) < 0 \\ &\implies f(x) < f(c)\,, \\ x \in (c, c + \delta) &\implies f(x) - f(c) = \frac{f(x) - f(c)}{x - c}(x - c) > 0 \\ &\implies f(x) > f(c)\,. \end{aligned}$$

□

5. [20] Let $f(x) = \log(1 + x^2)$ for every $x \in \mathbb{R}$. Show that

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, x^{2k} \quad \text{for every } x \in [-1, 1]\,,$$

and that the series diverges for all other $x \in \mathbb{R}$.

Partial Solution. It is easy to show that the series converges for every $x \in [-1, 1]$ and diverges otherwise.

The convergence when $|x| \le 1$ is best handled by the Alternating Series Test. Indeed, because the sequence

$$\left\{ \frac{1}{k}\, x^{2k} \right\}_{k=1}^{\infty} \quad \text{is decreasing and positive}\,.$$

and because

$$\lim_{k \to \infty} \frac{1}{k}\, x^{2k} = 0\,,$$

the Alternating Series Test shows that

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\, x^{2k} \quad \text{converges}\,.$$

The divergence when $|x| > 1$ follows from the Divergence Test because in that case

$$\lim_{k\to\infty} \frac{1}{k}\, x^{2k} = \infty \ (\neq 0)\,.$$

However, this is not a complete solution to the problem because these arguments do not show that when the series converges, it converges to $f(x)$. □

Solution. Because $x^2 \in [0,\infty)$ for every $x \in \mathbb{R}$, the problem reduces to showing that

$$\log(1+y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, y^k \quad \text{for every } y \in [0,1]\,,$$

and that the series diverges for every $y \in (1,\infty)$. The assertions will then follow upon setting $y = x^2$ into these results.

Let $g(y) = \log(1+y)$. By direct computation you see that

$$g'(y) = \frac{1}{1+y}\,, \quad g''(y) = \frac{-1}{(1+y)^2}\,, \quad g'''(y) = \frac{2}{(1+y)^3}\,, \quad g''''(y) = \frac{-6}{(1+y)^4}\,.$$

This should suggest to you that for every $k \in \mathbb{Z}_+$ one has

$$g^{(k)}(y) = (-1)^{k+1} \frac{(k-1)!}{(1+y)^k}\,,$$

which is easily verified by induction. Because $g(0) = 0$ while $g^{(k)}(0) = (-1)^{k+1}(k-1)!$ for every $k \in \mathbb{Z}_+$, the Lagrange Remainder Theorem implies that for every $y > 0$ there exists $p \in (0,y)$ such that

$$\begin{aligned} g(y) &= \sum_{k=0}^{n} \frac{1}{k!} g^{(k)}(0)\, y^k + \frac{1}{(n+1)!} g^{(n+1)}(p)\, y^{n+1} \\ &= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\, y^k + \frac{(-1)^n}{n+1} \frac{1}{(1+p)^{n+1}}\, y^{n+1}\,. \end{aligned}$$

Because $g(y) = \log(1+y)$ while $p \in (0,y)$, it follows that

$$\left| \log(1+y) - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\, y^k \right| = \frac{1}{n+1} \frac{1}{(1+p)^{n+1}}\, y^{n+1} < \frac{1}{n+1}\, y^{n+1}\,.$$

For every $y \in [0,1]$ we thereby obtain the uniform estimate

$$\left| \log(1+y) - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\, y^k \right| < \frac{1}{n+1}\,.$$

Hence,

$$\log(1+y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, y^k \quad \text{uniformly over } y \in [0,1]\,.$$

The fact that the series diverges for every $y > 1$ follows from the Divergence Test because in that case

$$\lim_{k\to\infty} \frac{1}{k}\, y^k = \infty \ (\neq 0)\,.$$

The assertions follow by setting $y = x^2$ into the above results. □

6. [20] Determine all $a \in \mathbb{R}$ for which the following formal infinite series converge. Give your reasoning.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{a^n}{3^n \log(n)}$

Solution. The series converges for every $a \in [-3, 3)$ and diverges otherwise.

The cases $|a| < 3$ and $|a| > 3$ are best handled by the Ratio Test. Let $b_n = a^n/(3^n \log(n))$. Because

$$\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = \lim_{n\to\infty} \frac{\log(n+1)}{\log(n)} \frac{|a|}{3} = \frac{|a|}{3},$$

the Ratio Test then implies that this series converges absolutely for $|a| < 3$ and diverges for $|a| > 3$.

The case $a = -3$ is best handled by the Alternating Series Test. Indeed, because the sequence

$$\left\{ \frac{1}{\log(n)} \right\}_{n=2}^{\infty} \quad \text{is decreasing and positive}.$$

and because

$$\lim_{n\to\infty} \frac{1}{\log(n)} = 0,$$

the Alternating Series Test shows that

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{\log(n)} \quad \text{converges}.$$

The case $a = 3$ is best handled by Limit Comparison Test, say with the harmonic series. Indeed, because

$$\lim_{n\to\infty} \frac{\log(n)}{n} = 0,$$

and because the harmonic series

$$\sum_{n=2}^{\infty} \frac{1}{n} \quad \text{diverges},$$

the Limit Comparison Test shows that

$$\sum_{n=2}^{\infty} \frac{1}{\log(n)} \quad \text{diverges}.$$

Alternatively, one could treat this case with the Direct Comparison Test, the Integral Test, or the Cauchy 2^k Test. □

(b) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k}{k^4 + 1} \right)^a$

Solution. The series converges for every $a \in (\frac{1}{3}, \infty)$ and diverges otherwise.

This is best handled by Limit Comparison Test. Because

$$\frac{k}{k^4+1} \sim \frac{1}{k^3} \quad \text{as } k \to \infty\,,$$

one sees that the original series should be compared with the p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^{3a}}\,.$$

Indeed, because for every $a \in \mathbb{R}$ one has

$$\lim_{k\to\infty} \frac{\left(\dfrac{k}{k^4+1}\right)^a}{\dfrac{1}{k^{3a}}} = \lim_{k\to\infty} \left(\frac{k^4}{k^4+1}\right)^a = 1\,,$$

the Limit Comparison Test then implies that

$$\sum_{k=1}^{\infty} \left(\frac{k}{k^4+1}\right)^a \quad \text{converges} \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} \frac{1}{k^{3a}} \quad \text{converges}\,.$$

Because $p = 3a$ for the p-series, that series converges for $a \in (\frac{1}{3}, \infty)$ and diverges otherwise. The same is therefore true for the original series. □

7. [20] Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be Riemann integrable over $[a,b]$. Show that $f+g$ is Riemann integrable over $[a,b]$.

Solution. The shortest route to a proof uses the Lebesgue Theorem. Let D_f, D_g, and D_{f+g} denote the points in $[a,b]$ at which f, g, and $f+g$ respectively are discontinuous. It is clear that $D_{f+g} \subset D_f \cup D_g$ because $f+g$ is continuous at every point where both f and g are continuous. Because f and g are Riemann integrable over $[a,b]$, one direction of the Lebesgue Theorem implies that D_f and D_g have measure zero. Because the union of two measure zero sets also has measure zero, and because any subset of a measure zero set also has measure zero, it follows that $D_{f+g} \subset D_f \cup D_g$ has measure zero. The other direction of the Lebesgue Theorem then implies that $f+g$ is Riemann integrable over $[a,b]$. □

Alternative Solution. Let $\epsilon > 0$. Because f and g are Riemann integrable over $[a.b]$, the Darboux Theorem implies that there exist partitions P_ϵ^f and P_ϵ^g of $[a.b]$ such that

$$0 \leq U(f, P_\epsilon^f) - L(f, P_\epsilon^f) < \frac{\epsilon}{2}\,, \qquad 0 \leq U(g, P_\epsilon^g) - L(f, P_\epsilon^g) < \frac{\epsilon}{2}\,.$$

Set $P_\epsilon = P_\epsilon^f \vee P_\epsilon^g$. Then

$$U(f+g, P_\epsilon) \leq U(f, P_\epsilon) + U(g, P_\epsilon) \leq U(f, P_\epsilon^f) + U(g, P_\epsilon^g)\,,$$
$$L(f+g, P_\epsilon) \geq L(f, P_\epsilon) + L(g, P_\epsilon) \geq L(f, P_\epsilon^f) + L(g, P_\epsilon^g)\,.$$

Upon combining the above inequalities you find that

$$\begin{aligned} 0 &\leq U(f+g, P_\epsilon) - L(f+g, P_\epsilon) \\ &\leq U(f, P_\epsilon^f) - L(f, P_\epsilon^f) + U(g, P_\epsilon^g) - L(g, P_\epsilon^g) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,. \end{aligned}$$

Because such a P_ϵ can be found for every $\epsilon > 0$, the Darboux Theorem implies that $f+g$ is Riemann integrable. □

Remark. The second solution gives a lot more. It is just a few steps away from showing that the intergral of $f+g$ is the sum of the integrals of f and g.

8. [20] A function $f:[a,b]\to\mathbb{R}$ is said to be Hölder continuous of order $\alpha\in(0,1]$ if there exists a $C\in\mathbb{R}_+$ such that for every $x,y\in[a,b]$ one has

$$|f(x)-f(y)|<C\,|x-y|^\alpha\,.$$

Let $f:[a,b]\to\mathbb{R}$ be Hölder continuous of order $\alpha\in(0,1]$.

(a) Show that f is uniformly continuous over $[a,b]$.

Solution. Let $\epsilon>0$. Set $\delta=(\epsilon/C)^{\frac{1}{\alpha}}$. Then for every $x,y\in[a,b]$ we have

$$|x-y|<\delta \qquad\Longrightarrow\qquad |f(x)-f(y)|<C\,|x-y|^\alpha<C\,\delta^\alpha=\epsilon\,.$$

Hence, f is uniformly continuous over $[a,b]$. □

(b) Show that for every partition P of $[a,b]$ one has

$$0\le U(f,P)-L(f,P)<|P|^\alpha C\,(b-a)\,.$$

Solution. Let $P=[x_0,x_1,\cdots,x_n]$ be any partition of $[a,b]$. Then

$$0\le U(f,P)-L(f,P)=\sum_{k=1}^{n}\big(\overline{m}_k-\underline{m}_k\big)(x_k-x_{k-1})\,,$$

where

$$\overline{m}_k=\sup\{f(x)\,:\,x\in[x_{k-1},x_k]\}\,,\qquad \underline{m}_k=\inf\{f(x)\,:\,x\in[x_{k-1},x_k]\}\,.$$

Because f is continuous over each $[x_{k-1},x_k]$, by the Extreme-Value Theorem there exists points $\overline{x}_k$, $\underline{x}_k\in[x_{k-1},x_k]$ such that $\overline{m}_k=f(\overline{x}_k)$ and $\underline{m}_k=f(\underline{x}_k)$. The Hölder continuity of f the gives

$$\begin{aligned}0\le U(f,P)-L(f,P)&=\sum_{k=1}^{n}\big(f(\overline{x}_k)-f(\underline{x}_k)\big)(x_k-x_{k-1})\\&\le C\sum_{k=1}^{n}|\overline{x}_k-\underline{x}_k|^\alpha(x_k-x_{k-1})\,.\end{aligned}$$

Because $\overline{x}_k$, $\underline{x}_k\in[x_{k-1},x_k]$ you have

$$|\overline{x}_k-\underline{x}_k|\le x_k-x_{k-1}\le\max\{x_m-x_{m-1}\,:\,m=1,\cdots,n\}\equiv|P|\,,$$

whereby

$$\begin{aligned}0\le U(f,P)-L(f,P)&\le C\sum_{k=1}^{n}|P|^\alpha(x_k-x_{k-1})\\&=C\,|P|^\alpha\sum_{k=1}^{n}(x_k-x_{k-1})=C\,|P|^\alpha\,(b-a)\,.\end{aligned}$$

□

9. [20] Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued functions over $D \subset \mathbb{R}$, and $\{M_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $|g_n(x)| \leq M_n$ for every $x \in D$ and $n \in \mathbb{N}$. Show that

$$\sum_{n=0}^{\infty} M_n < \infty \quad \Longrightarrow \quad \sum_{n=0}^{\infty} g_n(x) \quad \text{converges uniformly over } D\,.$$

Remark: You are being asked to prove the Weierstrauss M-Test.

Solution. The Absolute Comparison Test states that

$$\sum_{n=0}^{\infty} M_n < \infty \quad \Longrightarrow \quad \sum_{n=0}^{\infty} g_n(x) \quad \text{converges absolutely for every } x \in D\,.$$

You must show that this pointwise convergence is uniform. Equivalently, if you introduce

$$f_n(x) = \sum_{k=0}^{n} g_k(x)\,, \qquad f(x) = \sum_{k=0}^{\infty} g_k(x)\,,$$

then you must show that $f_n \to f$ uniformly over D.

Let $\epsilon > 0$. Because $\sum_{k=0}^{\infty} M_k < \infty$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$n > n_\epsilon \quad \Longrightarrow \quad \sum_{k=n+1}^{\infty} M_k < \epsilon\,.$$

Then for every $n > n_\epsilon$ and every $x \in D$ one has

$$\begin{aligned} \big|f_n(x) - f(x)\big| &= \left|\sum_{k=1}^{n} g_k(x) - \sum_{k=1}^{\infty} g_k(x)\right| = \left|\sum_{k=n+1}^{\infty} g_k(x)\right| \\ &\leq \sum_{k=n+1}^{\infty} |g_k(x)| \leq \sum_{k=n+1}^{\infty} M_k < \epsilon\,. \end{aligned}$$

Hence, $f_n \to f$ uniformly over D. □

10. [40] For each $n \in \mathbb{Z}_+$ define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = n^2 x e^{-nx}$.

(a) Sketch the graph of a typical f_n over $[0,1]$.

Solution. Because $f_n'(x) = n^2(1-nx)e^{-nx}$ is positive over $[0,\frac{1}{n})$ and negative over $(\frac{1}{n},1]$ for any $n > 1$ your sketch should show that:

- the value of $f_n(x)$ increases over $[0,\frac{1}{n}]$ from $f_n(0) = 0$ at $x = 0$ to a maximum of $f_n(\frac{1}{n}) = \frac{n}{e}$ at $x = \frac{1}{n}$;
- the value of $f_n(x)$ decreases over $[\frac{1}{n},1]$ from its maximum of $f_n(\frac{1}{n}) = \frac{n}{e}$ at $x = \frac{1}{n}$ to $f_n(1) = n^2 e^{-n}$ at $x = 1$.

This shows that as n increases the maximum value of f_n increases as $\frac{n}{e}$ while its location moves closer to $x = 0$. This understanding is helpful for the rest of the problem. □

(b) Show that $f_n \to 0$ pointwise over $[0, 1]$.

Solution. Because $f_n(0) = 0$ for every $n \in \mathbb{N}$, the convergence of $\{f_n(x)\}$ when $x = 0$ is obvious. Now consider the sequence $\{f_n(x)\}$ for $x \in (0, 1]$. By l'Hôpital applied twice to the $\frac{\infty}{\infty}$ indeterminant form we see that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^2 x}{e^{nx}} = \lim_{n\to\infty} \frac{2nx}{xe^{nx}} = \lim_{n\to\infty} \frac{2x}{x^2 e^{nx}} = 0\,.$$

Notice that the above derivatives are taken with respect to the variable n, which we consider extended to all $\mathbb{R}$ when applying l'Hôpital. Alternatively, rather than applying l'Hôpital you could argue that the limit vanishes because $e^{nx} \to \infty$ faster than $n^2 x \to \infty$ when $x > 0$. □

(c) Show that the limit in part (b) is not uniform over $[0, 1]$.

Solution. You must show that there exists $\epsilon > 0$ such that for every $m \in \mathbb{N}$ there exists $n > m$ and $x \in [0, 1]$ such that $f_n(x) \geq \epsilon$. This is easy to do. In fact, for any $\epsilon > 0$ one has for every $n > \epsilon e$ that $f_n(\frac{1}{n}) = \frac{n}{e} > \epsilon$. □

(d) For every $\delta > 0$ show that $f_n \to 0$ uniformly over $[\delta, 1]$.

Solution. Let $\epsilon > 0$. You must show there exists $n_\epsilon \in \mathbb{N}$ such that for every $n > n_\epsilon$ and $x \in [\delta, 1]$ one has $0 < f_n(x) < \epsilon$. Here are two approaches to doing this.

First Approach. First notice that for every $x \in [\delta, 1]$ and $n > 0$ one has the inequalities

$$0 < f_n(x) = n^2 x e^{-nx} < x^2 e^{-n\delta}\,.$$

Either by noticing that $x^2 e^{-n\delta} = f_n(\delta)/\delta$ and using the pointwise convergence of assertion (b), or by arguing as in the proof of assertion (b) one sees that

$$\lim_{n\to\infty} x^2 e^{-n\delta} = 0\,.$$

Hence, there exists $n_\epsilon \in \mathbb{N}$ such that for every $n > n_\epsilon$ one has $x^2 e^{-n\delta} < \epsilon$. It follows that for every $n > n_\epsilon$ and $x \in [\delta, 1]$ one has

$$0 < f_n(x) \leq x^2 e^{-n\delta} < \epsilon\,.$$

Therefore $f_n \to 0$ uniformly over $[\delta, 1]$. □

Second Approach. First notice from part (a) that if $\frac{1}{n} < \delta$ then f_n is decreasing over $[\delta, 1]$. Hence, for every $n > 1/\delta$ and $x \in [\delta, 1]$ one has

$$0 < f_n(x) \leq f_n(\delta)\,.$$

By the pointwise convergence of assertion (b) applied to the point $x = \delta$ there exists $n_\epsilon \in \mathbb{N}$ such that $n_\epsilon > 1/\delta$ and that for every $n > n_\epsilon$ one has $f_n(\delta) < \epsilon$. It follows that for every $n > n_\epsilon$ and $x \in [\delta, 1]$ one has

$$0 < f_n(x) \leq f_n(\delta) < \epsilon\,.$$

Therefore $f_n \to 0$ uniformly over $[\delta, 1]$. □

(e) Show that

$$\lim_{n\to\infty} \int_0^1 f_n = 1\,.$$

Solution. One integration by parts ($u = nx$, $v = -e^{-nx}$) yields

$$\begin{aligned}\int_0^1 f_n &= \int_0^1 n^2 x e^{-nx}\,\mathrm{d}x = -nxe^{-nx}\Big|_{x=0}^1 + \int_0^1 e^{-nx} n\,\mathrm{d}x \\ &= -ne^{-n} - e^{-nx}\Big|_{x=0}^1 = -ne^{-n} - e^{-n} + 1 = 1 - \frac{n+1}{e^n}\,.\end{aligned}$$

By l'Hôpital applied to the $\frac{\infty}{\infty}$ indeterminant form we see that

$$\lim_{n\to\infty} \frac{n+1}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0\,.$$

Hence,

$$\lim_{n\to\infty} \int_0^1 f_n = 1 - \lim_{n\to\infty} \frac{n+1}{e^n} = 1\,.$$

(f) Let $g : [0,1] \to \mathbb{R}$ be continuous. Show that

$$\lim_{n\to\infty} \int_0^1 f_n g = g(0)\,.$$

Solution. Assertion (e) implies that assertion (f) is equivalent to

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\big(g(x) - g(0)\big)\,\mathrm{d}x = 0\,.$$

But this will follow once we show that for every $\epsilon > 0$

$$\limsup_{n\to\infty} \int_0^1 f_n(x)\,\big|g(x) - g(0)\big|\,\mathrm{d}x \leq \epsilon\,.$$

Let $\epsilon > 0$. Because g is continous at 0, there exists $\delta > 0$ such that

$$x \in [0,\delta) \qquad \Longrightarrow \qquad \big|g(x) - g(0)\big| < \epsilon\,.$$

Because g is continuous over $[0,1]$, by the Extreme-Value Theorem it is bounded over $[0,1]$. Let $M = \sup\{|g(x)| \,:\, x \in [0,1]\}$. Then for every $n \in \mathbb{N}$

$$\begin{aligned}\int_0^1 f_n(x)\big|g(x) - g(0)\big|\,\mathrm{d}x &= \int_0^\delta f_n(x)\big|g(x) - g(0)\big|\,\mathrm{d}x + \int_\delta^1 f_n(x)\big|g(x) - g(0)\big|\,\mathrm{d}x \\ &\leq \int_0^\delta f_n(x)\,\epsilon\,\mathrm{d}x + \int_\delta^1 f_n(x)\,2M\,\mathrm{d}x \\ &\leq \epsilon \int_0^1 f_n(x)\,\mathrm{d}x + 2M \int_\delta^1 f_n(x)\,\mathrm{d}x\,.\end{aligned}$$

Assertion (e) and the uniform convergence of assertion (d) imply that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = 1\,, \qquad \lim_{n\to\infty} \int_\delta^1 f_n(x)\,\mathrm{d}x = 0\,.$$

whereby the previous inequality implies that

$$\begin{aligned}\limsup_{n\to\infty}\int_0^1 f_n(x)\big|g(x)-g(0)\big|\,\mathrm{d}x &\leq \epsilon\lim_{n\to\infty}\int_0^1 f_n(x)\,\mathrm{d}x + 2M\lim_{n\to\infty}\int_\delta^1 f_n(x)\,\mathrm{d}x\\ &= \epsilon\,.\end{aligned}$$

But as argued above, because this holds for every $\epsilon > 0$, assertion (f) follows. $\square$