Fifth Homework: MATH 410
Due Tuesday, 6 October 2009

1. Prove the divergence assertion of Proposition 3.11 in the notes.
2. Prove Proposition 3.12 in the notes.
3. Consider the set
$$\left\{ x \in \mathbb{R} \,:\, \sum_{n=0}^{\infty} \frac{(4n)!}{n!} \, \frac{(2n)!}{(3n)!} \, x^n \quad \text{converges} \right\}.$$
Use the ratio test to prove that this set is an interval and find its endpoints.
4. Determine all $x, p \in \mathbb{R}$ for which the Fourier p-series
$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^p} \quad \text{converges}.$$
5. Let X be a field. Use the field axioms to show that if $x, y \in X$ then $\big(x^{-1}y\big)^{-1} = y^{-1}x$.
6. Let X be a field. Use the field axioms to show that if $x, y \in X$ then $(-x)(-y) = xy$.
7. Let A and B be closed subsets of $\mathbb{R}$. Show that $A \cap B$ and $A \cup B$ are closed.
8. Consider the real sequence $\{b_k\}_{k\in\mathbb{N}}$ given by
$$b_k = (-1)^k \left(3 + \frac{1}{(k+1)^2} \right) \quad \text{for every } k \in \mathbb{N},$$
where $\mathbb{N} = \{0, 1, 2, \cdots\}$.
   (a) Give the first three terms of the subsequence $\{b_{3k}\}_{k\in\mathbb{N}}$.
   (b) Give the first three terms of the subsequence $\{b_{2^k-1}\}_{k\in\mathbb{N}}$.
   (c) Compute $\limsup_{k\to\infty} b_k$ and $\liminf_{k\to\infty} b_k$. Justify your answers.
9. Determine all the values of $a \in \mathbb{R}$ for which
$$\sum_{n=2}^{\infty} \frac{1}{\log(n)} \, a^n \quad \text{converges}.$$
10. Determine all the values of $a \in \mathbb{R}$ for which
$$\sum_{k=0}^{\infty} \left(\frac{2k+3}{k^4+1} \right)^a \quad \text{converges}.$$
11. Determine all the values of $a \in \mathbb{R}$ for which
$$\sum_{m=1}^{\infty} \frac{1}{m^2} \big(2 + (-1)^m\big)^m a^m \quad \text{converges}.$$
12. Let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let A be a subset of $\mathbb{R}$. Write the negations of the following assertions.
   (a) "For every $m \in \mathbb{R}$ one has $b_j > m$ frequently as $j \to \infty$."
   (b) "Every sequence in A has a subsequence that converges to a limit in A."