Fourteenth Homework: MATH 410
Due Tuesday, 8 December 2009

1. Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $g : [a, b] \to \mathbb{R}$ be Riemann integrable and nonnegative over $[a, b]$. Prove that if $\int_a^b g > 0$ then there exists a point $p \in (a, b)$ such that
$$\int_a^b fg = f(p) \int_a^b g\,.$$
(This strenghthens the integral mean-value theorem given as Theorem 3.3 in the notes.)
2. When $q \in \mathbb{N}$ the binomial expansion yields
$$(1 + x)^q = \sum_{k=0}^{n} \frac{q!}{k!(q-k)!}\, x^k = 1 + \sum_{k=1}^{q} \frac{q(q-1)\cdots(q-k+1)}{k!}\, x^k\,.$$
Now let $q \in \mathbb{R} - \mathbb{N}$. Let $f(x) = (1 + x)^q$ for every $x > -1$. Then
$$f^{(k)}(x) = q(q-1)\cdots(q-k+1)(1+x)^{q-k} \quad \text{for every } x > -1 \text{ and } k \in \mathbb{Z}_+\,.$$
The formal Taylor series of f about 0 is therefore
$$1 + \sum_{k=1}^{\infty} \frac{q(q-1)\cdots(q-k+1)}{k!}\, x^k\,.$$
Show that this series converges absolutely to $(1 + x)^q$ when $|x| < 1$ and diverges when $|x| > 1$. (This formula is Newton's extension of the binomial expansion to powers q that are real.)
3. Show that for every $q > -1$ one has
$$2^q = 1 + \sum_{k=1}^{\infty} \frac{q(q-1)\cdots(q-k+1)}{k!}\,,$$
while for every $q \leq -1$ the above series diverges. (Hint: This is the case $x = 1$ for the series in the previous problem.)
4. Exercise 1 of Section 9.2 in the text.
5. Exercise 4 of Section 9.2 in the text.
6. Exercise 6 of Section 9.2 in the text.
7. Exercise 1 of Section 9.3 in the text.
8. Exercise 4 of Section 9.3 in the text.
9. Exercise 6 of Section 9.3 in the text.
10. Exercise 3 of Section 9.4 in the text.
11. Exercise 4 of Section 9.4 in the text.
12. Let $g : [0, 1] \to \mathbb{R}$ be continuous. Show that
$$\lim_{n\to\infty} \int_0^1 n x^{n-1} g(x)\, \mathrm{d}x = g(1)\,.$$