Fifteenth Homework: MATH 410
Due Tuesday, 15 December 2009

1. Exercise 2 of Section 8.6 in the text.
2. Exercise 3 of Section 8.6 in the text.
3. Exercise 1 of Section 9.5 in the text.
4. Exercise 4 of Section 9.5 in the text.
5. Exercise 10 of Section 9.5 in the text.
6. Let $[a, b] \subset \mathbb{R}$ be a closed, bounded interval. Let $f : [a, b] \to [a, b]$. Suppose there exists an $M \in (0, 1)$ such that
$$|f(x) - f(y)| \leq M\,|x - y| \quad \text{for every } x, y \in [a, b]\,.$$
Let $x_0 \in [a, b]$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ by
$$x_{n+1} = f(x_n) \quad \text{for every } n \in \mathbb{N}\,.$$
Show that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. (Hint: Consider $x_{n+1} - x_n = f(x_n) - f(x_{n-1})$.)
7. Let $f : (a, b) \to \mathbb{R}$ be differentiable at a point $c \in (a, b)$ with $f'(c) > 0$. Show that there exists a $\delta > 0$ such that
$$x \in (c - \delta, c) \subset (a, b) \implies f(x) < f(c)\,,$$
$$x \in (c, c + \delta) \subset (a, b) \implies f(c) < f(x)\,,$$
8. Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that there exists $p \in (a, b)$ such that
$$f(p) = \frac{1}{e^b - e^a} \int_a^b f(x) e^x \,\mathrm{d}x\,.$$
9. Consider a function f defined by
$$f(x) = \sum_{k=0}^{\infty} \frac{1}{4^k} \sin(3^k x)\,,$$
for every $x \in \mathbb{R}$ for which the above series converges.
   (a) Show that f is defined for every $x \in \mathbb{R}$.
   (b) Show that f is continuously differentiable over $\mathbb{R}$ and that
$$f(x) = \sum_{k=0}^{\infty} \frac{3^k}{4^k} \cos(3^k x)\,.$$
10. Let $f : [-1, 1] \to \mathbb{R}$ be continuous. Prove that
$$\lim_{n\to\infty} \int_{-1}^{1} \frac{n f(x)}{1 + n^2 x^2} = \pi f(0)\,.$$
11. Let $f(x) = \cosh(x)$ for every $x \in \mathbb{R}$. Then for every $k \in \mathbb{N}$ and every $x \in \mathbb{R}$ one has
$$f^{(2k)}(x) = \cosh(x)\,, \qquad f^{(2k+1)}(x) = \sinh(x)\,.$$
Show that
$$\cosh(x) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} \quad \text{for every } x \in \mathbb{R}\,,$$
and that the series converges uniformly over every interval of the form $[-R, R]$.
12. Prove that every countable set has measure zero.