

Eigenpairs and Diagonalizability
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1. EIGENPAIRS

Let A be an $n \times n$ matrix. A number λ (possibly complex even when A is real) is an *eigenvalue* of A if there exists a nonzero vector \mathbf{v} (possibly complex) such that

$$(1) \quad A\mathbf{v} = \lambda\mathbf{v}.$$

Each such vector is an *eigenvector* associated with λ , and (λ, \mathbf{v}) is an *eigenpair* of A .

Fact 1: If (λ, \mathbf{v}) is an eigenpair of A then so is $(\lambda, \alpha\mathbf{v})$ for every complex $\alpha \neq 0$. In other words, if \mathbf{v} is an eigenvector associated with an eigenvalue λ of A then so is $\alpha\mathbf{v}$ for every complex $\alpha \neq 0$. In particular, eigenvectors are not unique.

Reason. Because (λ, \mathbf{v}) is an eigenpair of A you know that (1) holds. It follows that

$$A(\alpha\mathbf{v}) = \alpha A\mathbf{v} = \alpha\lambda\mathbf{v} = \lambda(\alpha\mathbf{v}).$$

Because the scalar α and vector \mathbf{v} are nonzero, the vector $\alpha\mathbf{v}$ is also nonzero. Therefore $(\lambda, \alpha\mathbf{v})$ is also an eigenpair of A . □

1.1. **Finding Eigenvalues.** The characteristic polynomial of A is defined by

$$(2) \quad p_A(z) = \det(zI - A).$$

It has the form

$$p_A(z) = z^n + \pi_1 z^{n-1} + \pi_2 z^{n-2} + \cdots + \pi_{n-1} z + \pi_n,$$

where the coefficients $\pi_1, \pi_2, \dots, \pi_n$ are real if A is real. In other words, it is a monic polynomial of degree n . One can show that in general

$$\pi_1 = -\operatorname{tr}(A), \quad \pi_n = (-1)^n \det(A).$$

In particular, when $n = 2$ one has

$$p_A(z) = z^2 - \operatorname{tr}(A)z + \det(A).$$

Sometimes you will see the characteristic polynomial defined as $\det(A - zI)$. Because $\det(zI - A) = (-1)^n \det(A - zI)$, these two definitions of $p_A(z)$ coincides when n is even, and negatives of each other when n is odd. Both conventions are common. We have chosen the convention that makes $p_A(z)$ monic. What matters most about $p_A(z)$ is its roots and their multiplicity, which are the same for both conventions.

Fact 2: A number λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$. In other words, the eigenvalues of A are the roots of $p_A(z)$.

Reason. If λ is an eigenvalue of A then by (1) there exists a nonzero vector \mathbf{v} such that

$$(\lambda I - A)\mathbf{v} = \lambda\mathbf{v} - A\mathbf{v} = 0.$$

It follows that $p_A(\lambda) = \det(\lambda I - A) = 0$.

Conversely, if $p_A(\lambda) = \det(\lambda I - A) = 0$ then there exists a nonzero vector \mathbf{v} such that $(\lambda I - A)\mathbf{v} = 0$. It follows that

$$\lambda\mathbf{v} - A\mathbf{v} = (\lambda\mathbf{I} - A)\mathbf{v} = 0,$$

whereby λ and \mathbf{v} satisfy (1), which implies λ is an eigenvalue of A . □

Fact 2 shows that the eigenvalues of a $n \times n$ matrix A can be found if you can find all the roots of the characteristic polynomial of A . Because the degree of this characteristic polynomial is n , and because every polynomial of degree n has exactly n roots counting multiplicity, the $n \times n$ matrix A therefore must have at least one eigenvalue and at most n eigenvalues.

Example. Find the eigenvalues of $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

Solution. The characteristic polynomial of A is

$$p_A(z) = z^2 - 6z + 5 = (z - 1)(z - 5).$$

By Fact 2 the eigenvalues of A are 1 and 5.

Example. Find the eigenvalues of $A = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$.

Solution. The characteristic polynomial of A is

$$p_A(z) = z^2 - 6z + 13 = (z - 3)^2 + 4 = (z - 3)^2 + 2^2.$$

By Fact 2 the eigenvalues of A are $3 + i2$ and $3 - i2$.

Example. Find the eigenvalues of $A = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$.

Solution. The characteristic polynomial of A is

$$p_A(z) = z^2 - 6z + 9 = (z - 3)^2.$$

By Fact 2 the only eigenvalue of A is 3.

Remark. The above examples show all the possibilities that can arise for 2×2 real matrices — namely, a 2×2 real matrix A can have either, two real eigenvalues, a conjugate pair of eigenvalues, or a single real eigenvalue.

1.2. Finding Eigenvectors. Once you have found the eigenvalues of a matrix you can find all the eigenvectors associated with each eigenvalue by finding a general nonzero solution of (1).

You can quickly find the eigenvectors for any 2×2 matrix A with help from the Cayley-Hamilton Theorem, which states that $p_A(A) = 0$. The eigenvalues λ_1 and λ_2 are the roots of $p_A(z)$, so $p_A(z) = (z - \lambda_1)(z - \lambda_2)$. Hence, by the Cayley-Hamilton Theorem

$$(3) \quad 0 = p_A(A) = (A - \lambda_1 I)(A - \lambda_2 I) = (A - \lambda_2 I)(A - \lambda_1 I).$$

It follows that every nonzero column of $A - \lambda_2 I$ is an eigenvector associated with λ_1 , and that every nonzero column of $A - \lambda_1 I$ is an eigenvector associated with λ_2 . This observation holds even in the case where $\lambda_1 = \lambda_2$.

Example. Find the eigenpairs of $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

Solution. We have already showed that the eigenvalues of A are 1 and 5. Compute

$$A - I = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad A - 5I = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix},$$

Every column of $A - 5I$ has the form

$$\alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{for some } \alpha \neq 0,$$

while every column of $A - I$ has the form

$$\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for some } \alpha \neq 0.$$

It follows from (3) that the eigenpairs of A are

$$\left(1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \quad \left(5, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

Example. Find the eigenpairs of $A = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$.

Solution. We have already showed that the only eigenvalue of A is 3. Compute

$$A - 3I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Every column of $A - 3I$ has the form

$$\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for some } \alpha \neq 0.$$

It follows from (3) that an eigenpair of A is

$$\left(3, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

Example. Find the eigenpairs of $A = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$.

Solution. We have already showed that the eigenvalues of A are $3 + i2$ and $3 - i2$. Compute

$$A - (3 + i2)I = \begin{pmatrix} -i2 & 2 \\ -2 & -i2 \end{pmatrix}, \quad A - (3 - i2)I = \begin{pmatrix} i2 & 2 \\ -2 & i2 \end{pmatrix}.$$

Every column of $A - (3 - i2)I$ has the form

$$\alpha \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{for some } \alpha \neq 0,$$

while every column of $A - (3 + i2)I$ has the form

$$\alpha \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{for some } \alpha \neq 0.$$

It follows from (3) that the eigenpairs of A are

$$\left(3 + i2, \begin{pmatrix} 1 \\ i \end{pmatrix}\right), \quad \left(3 - i2, \begin{pmatrix} 1 \\ -i \end{pmatrix}\right).$$

1.3. Real Matrices. Notice that in the above example the eigenvectors associated with $3 - i2$ are complex conjugates to those associated with $3 + i2$. This illustrates a particular instance of the following general fact about real matrices.

Fact 3: If (λ, \mathbf{v}) is an eigenpair of a real matrix A then so is $(\bar{\lambda}, \bar{\mathbf{v}})$.

Reason. Because (λ, \mathbf{v}) is an eigenpair of A you know by (1) that $A\mathbf{v} = \lambda\mathbf{v}$. Because A is real, the complex conjugate of this equation is

$$A\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}},$$

where $\bar{\mathbf{v}}$ is nonzero because \mathbf{v} is nonzero. It follows that $(\bar{\lambda}, \bar{\mathbf{v}})$ is an eigenpair of A . \square

The examples given in the previous section illustrate particular instances of the following general facts.

Fact 4: Let λ be an eigenvalue of the real matrix A . If λ is real then it has a real eigenvector. If λ is not real then none of its eigenvectors are real.

Reason. Let \mathbf{v} be any eigenvector associated with λ , so that (λ, \mathbf{v}) is an eigenpair of A . Let $\lambda = \mu + i\nu$ and $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ where μ and ν are real numbers and \mathbf{u} and \mathbf{w} are real vectors. One then has

$$A\mathbf{u} + iA\mathbf{w} = A\mathbf{v} = \lambda\mathbf{v} = (\mu + i\nu)(\mathbf{u} + i\mathbf{w}) = (\mu\mathbf{u} - \nu\mathbf{w}) + i(\mu\mathbf{w} + \nu\mathbf{u}),$$

which is equivalent to

$$A\mathbf{u} - \mu\mathbf{u} = -\nu\mathbf{w}, \quad \text{and} \quad A\mathbf{w} - \mu\mathbf{w} = \nu\mathbf{u}.$$

If $\nu = 0$ then \mathbf{u} and \mathbf{w} will be real eigenvectors associated with λ whenever they are nonzero. But at least one of \mathbf{u} and \mathbf{w} must be nonzero because $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ is nonzero. Conversely, if $\nu \neq 0$ and $\mathbf{w} = 0$ then the second equation above implies $\mathbf{u} = 0$ too, which contradicts the fact that at least one of \mathbf{u} and \mathbf{w} must be nonzero. Hence, if $\nu \neq 0$ then $\mathbf{w} \neq 0$. \square

2. APPLICATION: SOLUTIONS OF FIRST-ORDER SYSTEMS.

We are now ready to use eigenvalues and eigenvectors to construct solutions of first-order differential systems with a constant coefficient matrix. The system we study is

$$(4) \quad \frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where $\mathbf{x}(t)$ is a vector and A is a real $n \times n$ matrix. We begin with the following basic fact.

Fact 5: If (λ, \mathbf{v}) is an eigenpair of A then a solution of (4) is

$$(5) \quad \mathbf{x}(t) = e^{\lambda t}\mathbf{v}.$$

Reason. By direct calculation we see that

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}(e^{\lambda t}\mathbf{v}) = e^{\lambda t}\lambda\mathbf{v} = e^{\lambda t}A\mathbf{v} = A(e^{\lambda t}\mathbf{v}) = A\mathbf{x},$$

whereby $\mathbf{x}(t)$ given by (5) solves (4). \square

If (λ, \mathbf{v}) is a real eigenpair of A then recipe (5) will yield a real solution of (4). But if λ is an eigenvalue of A that is not real then recipe (5) will not yield a real solution. However, if we also use the solution associated with the conjugate eigenpair $(\bar{\lambda}, \bar{\mathbf{v}})$ then we can construct two real solutions.

Fact 6: Let (λ, \mathbf{v}) be an eigenpair of A with $\lambda = \mu + i\nu$ and $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ where μ and ν are real numbers while \mathbf{u} and \mathbf{w} are real vectors. Then two real solutions of (4) are

$$(6) \quad \begin{aligned} \mathbf{x}_1(t) &= \operatorname{Re}(e^{\lambda t} \mathbf{v}) = e^{\mu t} (\mathbf{u} \cos(\nu t) - \mathbf{w} \sin(\nu t)), \\ \mathbf{x}_2(t) &= \operatorname{Im}(e^{\lambda t} \mathbf{v}) = e^{\mu t} (\mathbf{w} \cos(\nu t) + \mathbf{u} \sin(\nu t)). \end{aligned}$$

Reason. Because (λ, \mathbf{v}) is an eigenpair of A , by Fact 3 so is $(\bar{\lambda}, \bar{\mathbf{v}})$. By recipe (5) two solutions of (4) are $e^{\lambda t} \mathbf{v}$ and $e^{\bar{\lambda} t} \bar{\mathbf{v}}$, which are complex conjugates of each other. Because equation (4) is linear, it follows that two real solutions of (4) are given by

$$\mathbf{x}_1(t) = \operatorname{Re}(e^{\lambda t} \mathbf{v}) = \frac{e^{\lambda t} \mathbf{v} + e^{\bar{\lambda} t} \bar{\mathbf{v}}}{2}, \quad \mathbf{x}_2(t) = \operatorname{Im}(e^{\lambda t} \mathbf{v}) = \frac{e^{\lambda t} \mathbf{v} - e^{\bar{\lambda} t} \bar{\mathbf{v}}}{i2}.$$

Because $\lambda = \mu + i\nu$ and $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ we see that

$$\begin{aligned} e^{\lambda t} \mathbf{v} &= e^{\mu t} (\cos(\nu t) + i \sin(\nu t)) (\mathbf{u} + i\mathbf{w}) \\ &= e^{\mu t} [(\mathbf{u} \cos(\nu t) - \mathbf{w} \sin(\nu t)) + i(\mathbf{w} \cos(\nu t) + \mathbf{u} \sin(\nu t))], \end{aligned}$$

whereby $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are read off from the real and imaginary parts, yielding (6). \square

Example. Find two linearly independent real solutions of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \text{where } A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

Solution. By a previous example we know that A has the real eigenpairs

$$\left(1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \quad \left(5, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

By recipe (5) the equation has the real solutions

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_2(t) = e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

These solutions are linearly independent because

$$\det(\mathbf{x}_1(0) \quad \mathbf{x}_2(0)) = \det \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2 \neq 0.$$

Example. Find two linearly independent real solutions of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \text{where } A = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}.$$

Solution. By a previous example we know that A has the conjugate eigenpairs

$$\left(3 + i2, \begin{pmatrix} 1 \\ i \end{pmatrix}\right), \quad \left(3 - i2, \begin{pmatrix} 1 \\ -i \end{pmatrix}\right).$$

Because

$$e^{(3+i2)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{3t} (\cos(2t) + i \sin(2t)) \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{3t} \begin{pmatrix} \cos(2t) + i \sin(2t) \\ -\sin(2t) + i \cos(2t) \end{pmatrix},$$

by recipe (6) the equation has the real solutions

$$\mathbf{x}_1(t) = e^{3t} \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix}, \quad \mathbf{x}_2(t) = e^{3t} \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

These solutions are linearly independent because

$$\det(\mathbf{x}_1(0) \ \mathbf{x}_2(0)) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0.$$

3. DIAGONALIZABLE MATRICES

We call an $n \times n$ matrix A *diagonalizable* when there exists an invertible matrix V and a diagonal matrix D such that $A = VDV^{-1}$. To *diagonalize* A means to find such a V and D . We first show that A is diagonalizable if it has n linearly independent eigenvectors.

Fact 7: If a real $n \times n$ matrix A has n eigenpairs, $(\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2), \dots, (\lambda_n, \mathbf{v}_n)$, such that the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent then

$$(7) \quad A = VDV^{-1} \quad ,$$

where V is the $n \times n$ matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ — i.e.

$$(8) \quad V = (\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n) \ ,$$

while D is the $n \times n$ diagonal matrix

$$(9) \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} .$$

Reason. Underlying this result is the fact that

$$(10) \quad \begin{aligned} AV &= A(\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n) = (A\mathbf{v}_1 \ A\mathbf{v}_2 \ \cdots \ A\mathbf{v}_n) \\ &= (\lambda_1\mathbf{v}_1 \ \lambda_2\mathbf{v}_2 \ \cdots \ \lambda_n\mathbf{v}_n) \\ &= (\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n) \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} = VD . \end{aligned}$$

Once we show that V is invertible then (7) follows upon multiplying the above relation on the left by V^{-1} .

We claim that $\det(V) \neq 0$ because the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent. Suppose otherwise. Because $\det(V) = 0$ there exists a nonzero vector \mathbf{c} such that $V\mathbf{c} = 0$. This means that

$$0 = V\mathbf{c} = (\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n .$$

Because vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent, this implies $c_1 = c_2 = \cdots = c_n = 0$, which contradicts the fact \mathbf{c} is nonzero. Therefore $\det(V) \neq 0$. Hence, the matrix V is invertible and (7) follows upon multiplying relation (10) on the left by V^{-1} . \square

Fact 7 states that A is diagonalizable when it has n linearly independent eigenvectors. The converse of this statement is also true.

Fact 8: If a $n \times n$ matrix A is diagonalizable then it has n linearly independent eigenvectors.

Reason. Because A is diagonalizable it has the form $A = VDV^{-1}$ where the matrix V is invertible and the matrix D is diagonal.

Let the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be the columns of V . We claim these vectors are linearly independent. Indeed, if $0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$ then because $V = (\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n)$ we see that

$$0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n = (\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = V\mathbf{c}.$$

Because V is invertible, this implies that $\mathbf{c} = 0$. The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are therefore linearly independent.

Because $V = (\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n)$ and because $A = VDV^{-1}$ where D has the form (9), we see that

$$\begin{aligned} (A\mathbf{v}_1 \ A\mathbf{v}_2 \ \dots \ A\mathbf{v}_n) &= A(\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n) \\ &= AV = VDV^{-1}V = VD \\ &= (\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n) \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix} \\ &= (\lambda_1\mathbf{v}_1 \ \lambda_2\mathbf{v}_2 \ \dots \ \lambda_n\mathbf{v}_n). \end{aligned}$$

Because the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent, they are all nonzero. It then follows from the above relation that $(\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2), \dots, (\lambda_n, \mathbf{v}_n)$ are eigenpairs of A , such that the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent. \square

Example. Show that $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ is diagonalizable, and diagonalize it.

Solution. By a previous example we know that A has the real eigenpairs

$$\left(1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \quad \left(5, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

Because we also know the eigenvectors are linearly independent, A is diagonalizable. Then (8) and (9) yield

$$V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}.$$

Because $\det(V) = 2$, one has

$$V^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

It follows from (7) that A is diagonalized as

$$A = VDV^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Remark. While not every matrix is diagonalizable, most matrices are. Here we give four criteria that insure a real $n \times n$ matrix A is diagonalizable.

- If A has n distinct eigenvalues then it is diagonalizable.
- If A is Hermitian ($A^H = A$) then its eigenvalues are real ($\overline{\lambda_j} = \lambda_j$), and it will have n real eigenvectors \mathbf{v}_j that can be normalized so that $\mathbf{v}_j^H \mathbf{v}_k = \delta_{jk}$. With this normalization $V^{-1} = V^H$.
- If A is skew-Hermitian ($A^H = -A$) then its eigenvalues are imaginary ($\overline{\lambda_j} = -\lambda_j$), and it will have n eigenvectors \mathbf{v}_j that can be normalized so that $\mathbf{v}_j^H \mathbf{v}_k = \delta_{jk}$. With this normalization $V^{-1} = V^H$.
- If A is normal ($A^H A = A A^H$) then it will have n eigenvectors \mathbf{v}_j that can be normalized so that $\mathbf{v}_j^H \mathbf{v}_k = \delta_{jk}$. With this normalization $V^{-1} = V^H$.

Matrices that are either symmetric or skew-symmetric are also normal. There are normal matrices that are neither symmetric nor skew-symmetric. Because the normal criterion is harder to verify than the symmetric and skew-symmetric criteria, it should be checked last. Both of the examples we have given above have distinct eigenvalues. The first example is symmetric. The second is normal, but is neither symmetric nor skew-symmetric.