First In-Class Exam Solutions
Math 220, Professor David Levermore
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(1) [20] Compute the derivatives of the following functions.

(a) $f(x) = x^3 - 3x^2 + 5x + \dfrac{7}{x^2}$

Solution. Because $\dfrac{7}{x^2} = 7x^{-2}$ you see that

$$f'(x) = 3x^2 - 3 \cdot 2x + 5 + 7 \cdot \frac{-2}{x^3} = 3x^2 - 6x + 5 - \frac{14}{x^3}\,.$$

(b) $g(r) = \sqrt{1 + r^2}$

Solution. Because $g(r) = (1 + r^2)^{\frac{1}{2}}$ you see that

$$g'(r) = \tfrac{1}{2}(1 + r^2)^{-\frac{1}{2}} 2r = \frac{r}{\sqrt{1 + r^2}}\,.$$

(c) $h(s) = \dfrac{1}{2 + 3s}$

Solution. Because $h(s) = (2 + 3s)^{-1}$ you see that

$$h'(s) = -1(2 + 3s)^{-2} 3 = \frac{-3}{(2 + 3s)^2}\,.$$

(d) $j(t) = t^2(t + 2)$

Solution. Because $j(t) = t^3 + 2t^2$ you see that

$$j'(t) = 3t^2 + 2 \cdot 2t = 3t^2 + 4t\,.$$

(2) [10] Let $f(x) = 1/x$ and $g(x) = x^2 + 4$.

(a) Evaluate $g(x) - f(x)$.

Solution.

$$g(x) - f(x) = x^2 + 4 - \frac{1}{x}\,.$$

(b) Evaluate $f(g(x))$.

Solution.

$$f(g(x)) = \frac{1}{g(x)} = \frac{1}{x^2 + 4}\,.$$

(3) [15] Consider the function $g(x) = x^2 - 3x + 4$.

(a) Find the line tangent to the graph of g at $x = 3$.

Solution. The line tangent to the graph of g at $x = 3$ goes through the point $(3, g(3))$ and has slope $g'(3)$. Because $g(x) = x^2 - 3x + 4$ and $g'(x) = 2x - 3$, one sees that

$$g(3) = 3^2 - 3 \cdot 3 + 4 = 9 - 9 + 4 = 4\,,$$
$$g'(3) = 2 \cdot 3 - 3 = 6 - 3 = 3\,.$$

The point-slope form of this line can be expressed either as

$$y = g(3) + g'(3)(x - 3) = 4 + 3(x - 3) = 3x - 5\,,$$

or as

$$y - 4 = 3(x - 3)\,.$$

(b) Find the point x where the line tangent to the graph of g is parallel to the line $y = 6x - 7$.

Solution. Lines are parallel when they have the same slope. Because the slope of the line tangent to the graph of g at x is given by $g'(x) = 2x - 3$, while the line $y = 6x - 7$ has slope 6, you must find x such that $g'(x) = 2x - 3 = 6$. The point where this happens is $x = \frac{9}{2}$.

Remark. You were not asked to give the equation of the associated tangent line, but many of you did. Because

$$g(\tfrac{9}{2}) = \left(\tfrac{9}{2}\right)^2 - 3 \cdot \tfrac{9}{2} + 4 = \tfrac{81}{4} - \tfrac{27}{2} + 4 = \tfrac{81-54+16}{4} = \tfrac{43}{4}\,,$$

the equation of this line can be expressed either as

$$y = \tfrac{43}{4} + 6\big(x - \tfrac{9}{2}\big) = 6x - \tfrac{65}{4}\,,$$

or as

$$y - \tfrac{43}{4} = 6\big(x - \tfrac{9}{2}\big)\,.$$

(4) [20] Consider the function $f(x) = x^4 - 6x^2 + 8x + 12$. Its derivative is

$$f'(x) = 4x^3 - 12x + 8 = 4(x+2)(x-1)^2 \,.$$

(a) Identify the intervals over which f is increasing and decreasing.
(b) Identify the relative maximums and minimums.
(c) Identify the intervals over which f is concave up and concave down.
(d) Identify the upward and downward inflection points.

Solution (a). The critical points of f are found by setting $f'(x) = 0$. Because $f'(x) = 4(x+2)(x-1)^2$, the critical points are therefore $x = -2$ and $x = 1$.

Next, a sign analysis of $f'(x)$ shows that

- $f'(x) < 0$ over $(-\infty, -2)$ because $f'(-3) = 4(-1)(-4)^2 = -64$, or because $f'(x)$ behaves like $4x^3$, which is negative, as $x \to -\infty$;
- $f'(x) > 0$ over $(-2, 1)$ because $f'(0) = 4(2)(-1)^2 = 8$;
- $f'(x) > 0$ over $(1, \infty)$ because $f'(2) = 4(4)(1)^2 = 16$, or because $f'(x)$ behaves like $4x^3$, which is positive, as $x \to \infty$.

You can read-off from this sign analysis that

$$f \text{ is decreasing over } (-\infty, -2)\,,$$
$$f \text{ is increasing over } (-2, \infty)\,.$$

Solution (b). The critical point $x = -2$ is a relative minimum because $f'(x)$ changes sign from negative to positive. The critical point $x = 1$ is a not a relative extreme point because $f'(x)$ is positive on both sides of $x = 1$. Hence,

$$x = -2 \text{ is a relative minimum and there is no relative maximum}\,.$$

Remark. The second derivative test for extreme points fails to give any information at $x = 1$ because $f''(1) = 0$, so it cannot be used at this point.

Remark. The point $x = -2$ is an absolute minimum because f is decreasing everywhere to its left and increasing everywhere to its right.

Solution (c). The degenerate points of f are found by setting $f''(x) = 0$. Because $f''(x) = 12x^2 - 12 = 12(x^2-1) = 12(x+1)(x-1)$, the degenerate points are therefore $x = -1$ and $x = 1$.

Next, a sign analysis of $f''(x)$ shows that

- $f''(x) > 0$ over $(-\infty, -1)$ because $f''(-2) = 12(-1)(-3) = 36$ or because $f''(x)$ behaves like $12x^2$, which is positive, as $x \to -\infty$;
- $f''(x) < 0$ over $(-1, 1)$ because $f''(0) = 12(1)(-1) = -12$;
- $f''(x) > 0$ over $(1, \infty)$ because $f''(2) = 12(3)(1) = 36$ or because $f''(x)$ behaves like $12x^2$, which is positive, as $x \to \infty$.

You can read-off from this sign analysis that

$$f \text{ is concave up over } (-\infty, -1) \text{ and } (1, \infty)\,,$$
$$f \text{ is concave down over } (-1, 1)\,.$$

Solution (d). The degenerate point $x = -1$ is a downward inflection because $f''(x)$ changes sign from positive to negative. The degenerate point $x = 1$ is an upward inflection because $f''(x)$ changes sign from negative to positive. Hence,

$$x = -1 \text{ is a downward inflection and } x = 1 \text{ is an upward inflection}\,.$$

(5) [15] Let $f(p)$ be the number (in thousands) of computers sold when they are sold at p dollars each. Interpret the statements $f(1200) = 60$, and $f'(1200) = -.02$. Estimate the number of computers sold if the price is set at 1250 dollars each.

Solution. The statement $f(1200) = 60$ means

60,000 computers will be sold when they are sold at 1200 dollars each.

The statement $f'(1200) = -.02$ means

20 less computers will be sold for each dollar the price is raised.

If the price is set at 1250 dollars each then the number of computers sold (in thousands) can be approximated by

$$f(1250) \approx f(1200) + f'(1200)(1250 - 1200) = 60 - .02 \cdot 50 = 60 - 1 = 59\,.$$

The number of computers sold if the price is set at 1250 dollars each is approximately 59,000.

(6) [20] Sketch the graph of the function $f(x) = x + \dfrac{4}{x^2} - 3$ over $x > 0$. Label all extreme points and inflections. Your sketch should show the monotonicity and concavity of the function.

Solution. The critical points are found by solving $f'(x) = 0$ over $x > 0$. Because

$$f'(x) = 1 - \frac{8}{x^3} = \frac{x^3 - 8}{x^8}\,,$$

The only critical point in $x > 0$ is where $x^3 = 8$, which is $x = 2$.

Next, a sign analysis of $f'(x)$ over $(0, \infty)$ shows that

- $f'(x)$ is negative over $(0, 2)$ because $f'(1) = 1 - 8 = -7$ or because $f'(x)$ behaves like $-8/x^3$ as $x \to 0$, which is negative for $x > 0$;
- $f'(x)$ is positive over $(2, \infty)$ because $f'(4) = 1 - \frac{8}{64} = 1 - \frac{1}{8} = \frac{7}{8}$, or because $f'(x)$ behaves like x as $x \to \infty$, which is positive.

You can read-off from this sign analysis that

$$f \text{ is decreasing over } (0, 2)\,,$$
$$f \text{ is increasing over } (2, \infty)\,.$$

Moreover, $x = 2$ is the absolute minimum of f over $x > 0$ because f is decreasing everywhere to its left and increasing everywhere to its right. Hence, the minimum value of f is $f(2) = 2 + 1 - 3 = 0$.

Because $f''(x) = \dfrac{24}{x^4} > 0$ when $x > 0$, f is concave up over $x > 0$.

Your graph should have a minimum at the point $(2, 0)$, be concave up, and blow up as $x \to 0$ from the right and as $x \to \infty$.