Quiz 5 Solutions, Math 220, Professor David Levermore
Wednesday, 29 September 2010

(1) [5] Consider the function $f(x) = \frac{1}{3}x^3 - x^2 + 1$. Its derivative is $f'(x) = x^2 - 2x$.
(a) Find all the critical points of $f(x)$.
(b) Identify the intervals over which $f(x)$ is increasing and decreasing.
(c) Identify the relative minimums and relative maximums of $f(x)$.

Solution (a). The critical points of $f(x)$ are found by solving $f'(x) = 0$. Because $f'(x) = x^2 - 2x = x(x-2)$, the critical points are $x = 0$ and $x = 2$.

Solution (b). Because $f'(-1) = 1+2 = 3$, you see that $f'(x) > 0$ for $= \infty < x < 0$. Because $f'(1) = 1 - 2 = -1$, you see that $f'(x) < 0$ for $0 < x < 2$. Because $f'(3) = 9 - 6 = 3$, you see that $f'(x) > 0$ for $2 < x < \infty$. Hence,

$$f \text{ is increasing over } = \infty < x < 0 \text{ and over } 2 < x < \infty\,;$$
$$f \text{ is decreasing over } 0 < x < 2\,.$$

Solution (c). The critical point $x = 0$ is a relative maximum because F is increasing on its left and decreasing on its right. The critical point $x = 2$ is a relative minimum because F is decreasing on its left and increasing on its right.

(2) [5] Sketch the graph of the function $f(x) = x - 4 + \dfrac{9}{x}$ over $x > 0$. You can use the facts that $f'(x) = 1 - \dfrac{9}{x^2}$ and $f''(x) = \dfrac{18}{x^3}$.

Solution. The critical points are found by solving $f'(x) = 0$ over $x > 0$. Because

$$f'(x) = 1 - \frac{9}{x^2} = \frac{x^2 - 9}{x^2} = \frac{(x-3)(x+3)}{x^2}\,,$$

The only critical point in $x > 0$ is $x = 3$. Because $f'(1) = 1 - 9 = -8$, f is decreasing over $0 < x < 3$. Because $f'(6) = 1 - \frac{9}{36} = \frac{3}{4}$, f is increasing over $3 < x < \infty$. Hence, $x = 3$ is the absolute minimum of f over $x > 0$. Its value is $f(3) = 3 - 4 + 3 = 2$.

Because $f''(x) = \dfrac{18}{x^3} > 0$ when $x > 0$, f is concave up over $x > 0$.

Your graph should have a minimum at the point $(3, 2)$, be concave up, and blow up as x approaches 0 from the right and as x approaches ∞.