Quiz 15 Solutions, Math 220, Professor David Levermore
Friday, 10 December 2010

(1) [5] Twelve hundred square inches of material are available to construct a box with a square base, four rectangular sides, and no top. Find the dimensions of the box that maximize its volume.

Solution. Let x be the length of the edges along the bottom of the box, and y be the height of the box. The objective is to maximize $V = x^2 y$ subject to the constraint that $x^2 + 4xy = 1200$.

By Lagrange Multipliers. Set

$$F(x, y, \lambda) = x^2 y + \lambda(1200 - x^2 - 4xy)\,.$$

To find (x, y, λ) you must solve the three equations

$$0 = \frac{\partial F}{\partial x} = 2xy - 2\lambda x - 4\lambda y\,,$$
$$0 = \frac{\partial F}{\partial y} = x^2 - 4\lambda x = (x - 4\lambda)x\,,$$
$$0 = \frac{\partial F}{\partial \lambda} = 1200 - x^2 - 4xy\,.$$

By solving the second equation (which is the simplest) for x you find that $x = 4\lambda$. By eliminating x from the first equation you see that

$$0 = 2(4\lambda)y - 2\lambda(4\lambda) - 4\lambda y = 4\lambda(y - 2\lambda)\,.$$

By solving this equation for y you find that $y = 2\lambda$. By eliminating both x and y from the third equation you see that

$$1200 = (4\lambda)^2 + 4(4\lambda)(2\lambda) = 16\lambda^2 + 32\lambda^2 = 48\lambda^2\,.$$

By solving this equation for λ you find that $\lambda^2 = \frac{1200}{48} = \frac{100}{4} = 25$, which gives $\lambda = \pm 5$. Because x and y must be positive,

$$\lambda = 5\,, \qquad x = 4\lambda = 4 \cdot 5 = 20\,, \qquad y = 2\lambda = 2 \cdot 5 = 10\,.$$

The box should have 20 inch edges along its base and a 10 inch height.

By Earlier Method. By solving the constraint equation for y you find that

$$y = \frac{1200 - x^2}{4x}\,.$$

The objective then becomes

$$V(x) = x^2 \frac{1200 - x^2}{4x} = x\frac{1200 - x^2}{4} = 300x - \frac{x^3}{4}\,.$$

To find x you must solve

$$0 = V'(x) = 300 - \tfrac{3}{4}x^2 = \tfrac{3}{4}(400 - x^2)\,.$$

This leads to $x = 20$, whereby

$$y = \frac{1200 - 20^2}{4 \cdot 20} = \frac{1200 - 400}{80} = \frac{800}{80} = 10\,.$$

The box should have 20 inch edges along its base and a 10 inch height.

(2) [5] Use the method of least squares to fit a line $y = Ax + B$ to the data points $(0, 8)$, $(1, 7)$, $(2, 0)$.

Solution. The method of least squares is to minimize

$$\begin{aligned} \mathrm{Sq}(A, B) &= \Sigma_{i=1}^{3}(Ax_i + B - y_i)^2 \\ &= (A \cdot 0 + B - 8)^2 + (A \cdot 1 + B - 7)^2 + (A \cdot 2 + B - 0)^2 \\ &= (B - 8)^2 + (A + B - 7)^2 + (2A + B)^2 \,. \end{aligned}$$

To find A and B you must solve the two equations

$$\begin{aligned} 0 = \frac{\partial\,\mathrm{Sq}}{\partial A} &= 2(A + B - 7) + 2(2A + B)2 \\ &= 2A + 2B - 14 + 8A + 4B \\ &= 10A + 6B - 14 \,, \\ 0 = \frac{\partial\,\mathrm{Sq}}{\partial B} &= 2(B - 8) + 2(A + B - 7) + 2(2A + B) \\ &= 2B - 16 + 2A + 2B - 14 + 4A + 2B \\ &= 6A + 6B - 30 \,. \end{aligned}$$

By solving the second equation for B you find $B = 5 - A$. By eliminating B from the first equation you see that

$$0 = 10A + 6(5 - A) - 14 = 10A + 30 - 6A - 14 = 4A + 16 \,.$$

By solving this equation for A you find that $A = -4$. When this is placed into the equation for B you find $B = 5 - (-4) = 5 + 4 = 9$. Therefore the line $y = -4x + 9$ gives the best fit.

Solution by Formula. The general formula is

$$\begin{aligned} A &= \frac{N\Sigma x_i y_i - (\Sigma x_i)(\Sigma y_i)}{N\Sigma x_i^2 - (\Sigma x_i)^2} \,, \\ B &= \frac{\Sigma y_i - A\Sigma x_i}{N} \,. \end{aligned}$$

In this problem $N = 3$,

$$\begin{aligned} \Sigma x_i y_i &= 0 \cdot 8 + 1 \cdot 7 + 2 \cdot 0 = 0 + 7 + 0 = 7 \,, \\ \Sigma x_i^2 &= 0^2 + 1^2 + 2^2 = 0 + 1 + 4 = 5 \,, \\ \Sigma x_i &= 0 + 1 + 2 = 3 \,, \\ \Sigma y_i &= 8 + 7 + 0 = 15 \,. \end{aligned}$$

Upon placing these values into the general formula you find

$$\begin{aligned} A &= \frac{3 \cdot 7 - 3 \cdot 15}{3 \cdot 5 - 3^2} = \frac{7 - 15}{5 - 3} = \frac{-8}{2} = -4 \,, \\ B &= \frac{15 - (-4)3}{3} = \frac{15 + 12}{3} = \frac{27}{3} = 9 \,. \end{aligned}$$

Therefore the line $y = -4x + 9$ gives the best fit.