MORE APPLICATIONS OF DERIVATIVES

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This is a review of material pertaining to local approximations and their applications that are covered sometime during a year-long calculus sequence. It covers tangent line approximations, Taylor approximations and expansions, how to quickly generate Taylor expansions, limits of indeterminant forms, and l'Hopital's rules.

1. TANGENT LINE APPROXIMATIONS

1.1: Tangent Line Approximations. If f is differentiable at a point c then recall that the tangent line to the curve $y = f(x)$ at c is given by

$$y = f(c) + f'(c)(x - c)\,. \tag{1.1}$$

It is the unique line through the point $(c, f(c))$ with slope $f'(c)$.

The idea of the tangent line approximation is that this line will be a good approximation to the curve $y = f(x)$ so long as x is close to c. Viewed graphically, this idea should seem obvious to you. Another way to understand the tangent line approximation starts with the definition of the derivative at c written in the form

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}\,. \tag{1.2}$$

This can be re-expressed as

$$\lim_{x \to c} \frac{f(x) - f(c) - f'(c)(x - c)}{x - c} = 0\,. \tag{1.3}$$

Because $f(x) - f(c) - f'(c)(x - c)$ is the difference between the value of $f(x)$ and the value of its tangent line approximation at c, this shows that error of the tangent line approximation goes to zero faster than $x - c$ as x approaches c.

The sign of the error made by the tangent line approximation can be determined by analyzing the concavity of f near the point c. Let f be differentiable over an interval I that contains c. If f is concave up over I then

$$f(x) \geq f(c) + f'(c)(x - c) \qquad \text{for every } x \text{ in } I\,. \tag{1.4}$$

Said another way, if f is concave up over I, the tangent line lies below the graph of f over I. On the other hand, if f is concave down over I then

$$f(x) \leq f(c) + f'(c)(x - c) \qquad \text{for every } x \text{ in } I\,. \tag{1.5}$$

Said another way, if f is concave down over I, the tangent line lies above the graph of f over I.

2. TAYLOR APPROXIMATIONS AND EXPANSIONS

2.1: Taylor Polynomial Approximations. Taylor approximations are extensions of the tangent line approximation. Recall that if a function f is differentiable at c then the tangent line (1.1) to the curve $y = f(x)$ at c is just the unique line through the point $(c, f(c))$ with slope $f'(c)$. In this spirit, if f is twice differentiable at c then the curve

$$y = f(c) + f'(c)(x - c) + \frac{1}{2} f''(c)(x - c)^2 \,, \tag{2.1}$$

goes through the point $(c, f(c))$ and matches the first two derivatives of f at c. It is a parabola whenever $f''(c) \neq 0$. In the same spirit, if f is thrice differentiable at c then the curve

$$y = f(c) + f'(c)(x - c) + \frac{1}{2} f''(c)(x - c)^2 + \frac{1}{6} f'''(c)(x - c)^3 \,, \tag{2.2}$$

goes through the point $(c, f(c))$ and matches the first three derivatives of f at c. It is cubic whenever $f'''(c) \neq 0$.

In general, if f is n times differentiable at a point c then the n^{th} order Taylor approximation to $f(x)$ at c is given by the polynomial curve

$$y = f(c) + f'(c)(x - c) + \frac{1}{2} f''(c)(x - c)^2 + \cdots + \frac{1}{n!} f^{(n)}(c)(x - c)^n \,. \tag{2.3}$$

The right-hand side is a polynomial in x of degree at most n. It is expressed in powers of $(x - c)$. Its degree will be n whenever $f^{(n)}(c) \neq 0$, otherwise it will be less than n. It is the unique polynomial that matches f and its first n derivatives evaluated at c. The idea of the Taylor approximation is that if you match more derivatives at c then the approximation will be better near c. The summation notation may be used to express (2.3) more compactly as

$$y = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(x - c)^k \,. \tag{2.4}$$

If you are not familiar with this notation, do not worry, because you can get along fine without it for now. If however you plan a technical career, I suggest that you take the time to become familiar with it because you will only see more of it.

An important special case is n^{th} order Taylor approximation to $f(x)$ at 0. Then (2.3) takes the cleaner looking form

$$y = f(0) + f'(0)\, x + \frac{1}{2} f''(0)\, x^2 + \cdots + \frac{1}{n!} f^{(n)}(0)\, x^n \,, \tag{2.5}$$

which in the summation notation looks like

$$y = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(0)\, x^k \,. \tag{2.6}$$

Given a function f that is n times differentiable at c, one computes the n^{th} order Taylor approximation at c in three steps:

(1) compute a list of the first n derivatives of f

$$f(x)\,, \quad f'(x)\,, \quad f''(x)\,, \quad \cdots\,, \quad f^{(n)}(x)\,; \tag{2.7}$$

(2) evaluate all these functions at c to get the list of $n+1$ *numbers*

$$f(c)\,, \quad f'(c)\,, \quad f''(c)\,, \quad \cdots\,, \quad f^{(n)}(c)\,; \tag{2.8}$$

(3) plug these numbers into the right-hand side of (2.3) to obtain the coefficients of the powers of $(x-c)$.

These three steps should always yield a polynomial expressed in powers of $(x-c)$ that is of at most degree n. It will have at most $n+1$ terms in it, but will have less when some of the numbers in (2.8) are zero.

Example: To compute the second order Taylor approximation of $f(x) = \tan(x)$ at $\frac{\pi}{4}$: you first compute the derivatives

$$f(x) = \tan(x)\,, \quad f'(x) = \sec^2(x)\,, \quad f''(x) = 2\tan(x)\sec^2(x)\,;$$

which you evaluate as

$$f(\tfrac{\pi}{4}) = 1\,, \qquad f'(\tfrac{\pi}{4}) = 2\,, \qquad f''(\tfrac{\pi}{4}) = 4\,;$$

whereby from (2.3) you obtain the parabola

$$y = 1 + 2(x - \tfrac{\pi}{4}) + 2(x - \tfrac{\pi}{4})^2\,.$$

Example: To compute the third order Taylor approximation of $f(x) = e^x$ at 0: you first compute the derivatives

$$f(x) = e^x\,, \quad f'(x) = e^x\,, \quad f''(x) = e^x\,, \quad f'''(x) = e^x\,;$$

which you evaluate as

$$f(0) = 1\,, \qquad f'(0) = 1\,, \qquad f''(0) = 1\,, \qquad f'''(0) = 1\,;$$

whereby from (2.5) you obtain the cubic

$$y = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3\,.$$

Example: To compute the seventh order Taylor approximation of $f(x) = \ln(x)$ at 1, we would first compute the derivatives of ln through seventh order as

$$\ln(x)\,, \quad \frac{1}{x}\,, \quad \frac{-1}{x^2}\,, \quad \frac{2}{x^3}\,, \quad \frac{-6}{x^4}\,, \quad \frac{24}{x^5}\,, \quad \frac{-120}{x^6}\,, \quad \frac{720}{x^7}\,.$$

These functions are then be evaluated at 1 to obtain the numbers

$$0\,, \quad 1\,, \ -1\,, \quad 2\,, \ -6\,, \quad 24\,, \ -120\,, \quad 720\,.$$

Finally, these numbers are plugged into (2.3) to obtain

$$\begin{aligned} y = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \\ + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 + \frac{1}{7}(x-1)^7\,. \end{aligned} \tag{2.9}$$

Remark: Notice that the values of a function f and its first n derivatives evaluated at c can read off from its n^{th} order Taylor approximation at c. That is, suppose you are told that f has the n^{th} order Taylor approximation

$$y = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n\,, \tag{2.10}$$

where $a_0, a_1, \cdots, a_n$ are given numbers. Upon comparing this with (2.3) you see that

$$f^{(k)}(c) = k!\, a_k \qquad \text{for } k = 0, 1, \cdots, n\,.$$

Moreover, all Taylor approximations at c of lower order can be read off from the n^{th} order Taylor approximation of f at c. That is, suppose you are told that f has the n^{th} order Taylor approximation (2.10). Then you know that the second and third order Taylor approximations of f at c are simply

$$\begin{aligned} y &= a_0 + a_1(x-c) + a_2(x-c)^2\,, \\ y &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3\,. \end{aligned}$$

Example: If you know that the seventh order Taylor approximation of ln at 1 is given by (2.9) then the first, third, and fifth order Taylor approximations at 1 are:

$$\begin{aligned} y &= (x-1)\,, \\ y &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3\,, \\ y &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5\,. \end{aligned}$$

Can you read off the lower order Taylor approximations of even order?

2.2: Formal Taylor Series. Because the n^{th} order Taylor approximation of f at c contains all the terms of the Taylor approximations of lower order, if f is infinitely differentiable at c then you may encode all the Taylor approximations of finite order at c as the formal infinite series

$$T_c f(x) = f(c) + f'(c)(x-c) + \frac{1}{2}f''(c)(x-c)^2 + \cdots + \frac{1}{n!}f^{(n)}(c)(x-c)^n + \cdots . \tag{2.11}$$

which in summation notation looks like

$$T_c f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(c)(x-c)^k . \tag{2.12}$$

This is called the formal Taylor series (or Taylor expansion) of f at c. The formal Taylor series of f at 0 is simply

$$Tf(x) = f(0) + f'(0)\,x + \frac{1}{2}f''(0)\,x^2 + \cdots + \frac{1}{n!}f^{(n)}(0)\,x^n + \cdots . \tag{2.13}$$

which in summation notation looks like

$$Tf(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0)\,x^k . \tag{2.14}$$

It should be emphasized that we are not claiming any of the above sums can be assigned a value at a point x where the sum has an infinite number of nonzero terms. Nor are we claiming that when such a value can be assigned at such an x, the value will be equal to $f(x)$. Indeed, these things are not true for most infinitely differentiable functions. In Calculus II you will in see however that for many of the basic elementary functions these things are true at least at some such x. Let us put these questions aside. For now consider formal Taylor series to be a bookkeeping device by which one encodes all the Taylor approximations of finite order at c for any given function f that is infinitely differentiable at c.

In order to compute a formal Taylor series directly from f, you must notice a pattern in the expressions for $f^{(n)}$, so that you can write down the general expression of the n^{th} derivative of f. This may not be so easy to do, but once it is done, you just evaluate this general expression at c and plug the result into either (2.11) or (2.12). Fortunately, this is easy to do for many basic elementary functions.

Example: To compute Te^x, we set $f(x) = e^x$ and notice that

$$f^{(n)}(x) = e^x \qquad \text{for every } n\,.$$

Hence, the evaluation of these functions at 0 leads to

$$f^{(n)}(0) = 1 \qquad \text{for every } n\,.$$

When these numbers are plugged into (2.13) you find

$$Te^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \cdots\,, \tag{2.15}$$

which agrees with (2.10). When (2.15) is expressed in summation notation (2.14) you find

$$Te^x = \sum_{k=0}^{\infty} \frac{1}{k!}\, x^k\,. \tag{2.16}$$

This VERY IMPORTANT expansion should be memorized.

Example: To compute $T\sin(x)$, we notice that successive derivatives of sin form the repeating pattern

$$\sin(x)\,, \quad \cos(x)\,, \quad -\sin(x)\,, \quad -\cos(x)\,, \quad \cdots\,.$$

The corresponding repeating pattern of numbers is

$$0\,, \quad 1\,, \quad 0\,, \quad -1\,, \quad \cdots\,.$$

Hence $\sin^{(n)}(0)$ will vanish for even values of n and alternate between 1 and -1 for odd values of n. When these numbers are plugged into (2.13) you find

$$T\sin(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots\,. \tag{2.17}$$

Notice the striking relation between this expansion and that for $\exp(x)$; the terms here are just the odd terms of (2.15) but taken with an alternating sign. This can be expressed in summation notation (2.14) as

$$T\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1}\,. \tag{2.18}$$

Do you see that only odd powers of x appear in this sum and that the sign of the terms alternates? Write out the first four terms of this sum (those corresponding to $k = 0, 1, 2, 3$) to see that they agree with those shown in (2.17).

Example: To compute $T\cosh(x)$, we notice that successive derivatives of cosh form the repeating pattern

$$\cosh(x)\,, \quad \sinh(x)\,, \quad \cdots\,.$$

The corresponding repeating pattern of numbers is

$$1\,, \quad 0\,, \quad \cdots\,.$$

Hence $\cosh^{(n)}(0)$ will take the value 1 for even values of n and vanish for odd values of n. When these numbers are plugged into (2.13) you find

$$T\cosh(x) = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \cdots\,. \tag{2.19}$$

Once again notice the striking relation between this expansion and that for $\exp(x)$; the terms here are just the even terms of (2.15). This can be expressed in summation notation (2.14) as

$$T\cosh(x) = \sum_{k=0}^{\infty} \frac{1}{(2k)!}\, x^{2k}\,. \tag{2.20}$$

Write out the first four terms of this sum (those corresponding to $k = 0, 1, 2, 3$) to see that they agree with those shown in (2.19).

Example: To compute $T_1 \ln(x)$, we notice that for $n \geq 1$ the n^{th} derivative of $\ln(x)$ has the form

$$(-1)^{n-1}\frac{(n-1)!}{x^n}\,.$$

The corresponding n^{th} number obtained by evaluating this at 1 is

$$(-1)^{n-1}(n-1)!\,.$$

Finally, these numbers are plugged into (2.11) to obtain

$$\begin{aligned} T_1 \ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \\ + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 + \frac{1}{7}(x-1)^7 - \cdots\,. \end{aligned} \tag{2.21}$$

This can be expressed in summation notation (2.12) as

$$T_1 \ln(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\,(x-1)^k\,. \tag{2.22}$$

Example: To compute $T_c x^p$ for some $c > 0$, we notice that for $n \geq 1$ the n^{th} derivative of x^p has the form

$$p(p-1)\cdots(p-n+1)x^{p-n}\,.$$

The corresponding n^{th} number obtained by evaluating this at c is

$$p(p-1)\cdots(p-n+1)c^{p-n}\,.$$

Finally, these numbers are plugged into (2.11) to obtain

$$\begin{aligned} T_c x^p = c^p + pc^{p-1}(x-c) + \frac{p(p-1)}{2}c^{p-2}(x-c)^2 + \frac{p(p-1)(p-2)}{6}c^{p-3}(x-c)^3 \\ - \frac{p(p-1)(p-2)(p-3)}{24}c^{p-4}(x-c)^4 + \cdots\,. \end{aligned} \tag{2.23}$$

This can be expressed in summation notation (2.12) as

$$T_c x^p = c^p + \sum_{k=1}^{\infty} \frac{p(p-1)\cdots(p-k+1)}{k!}\,c^{p-k}(x-c)^k\,. \tag{2.24}$$

Remark: Notice that when p is a positive integer the sum (2.24) will truncate after p terms. Moreover, observe that in that case

$$\frac{p(p-1)\cdots(p-k+1)}{k!} = \frac{p!}{k!(p-k)!}\,.$$

You should recognize this as a coefficient of the binomial expansion. Indeed, when p is a positive integer the Taylor expansion (2.24) reduces to your old friend the binomial expansion:

$$(c+z)^p = c^p + pc^{p-1}z + \frac{p(p-1)}{2}c^{p-2}z^2 + \cdots + pcz^{p-1} + z^p\,.$$

For example, if one sets $z = x - c$, so that $x = z + c$, then for $p = 4$ the binomial expansion yields

$$\begin{aligned} x^4 &= (c+z)^4 = c^4 + 4c^3z + 6c^2x^2 + 4cz^3 + z^4 \\ &= c^4 + 4c^3(x-c) + 6c^2(x-c)^2 + 4c(x-c)^3 + (x-c)^4\,. \end{aligned}$$

Because the sum on the right-hand side above agrees with that for $T_c x^4$ given by (2.24), we see that the Taylor expansion recovers x^4 exactly. The story is the same when p is any positive integer.

3. GENERATING TAYLOR APPROXIMATIONS

3.1: Some Special Taylor Expansions. Many Taylor approximations can be simply built up from the following seven basic Taylor expansions at 0, which you should know:

$$
\begin{aligned}
Te^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots + \frac{1}{n!}x^n + \cdots , \\
T\sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots , \\
T\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots , \\
T\sinh(x) &= x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots + \frac{1}{(2n+1)!}x^{2n+1} + \cdots , \\
T\cosh(x) &= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots + \frac{1}{(2n)!}x^{2n} + \cdots , \\
T\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots + \frac{(-1)^{n-1}}{n}x^n + \cdots , \\
T(1+x)^p &= 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{6}x^3 + \cdots \\
&\qquad \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \cdots .
\end{aligned}
\tag{3.1}
$$

When expressed in summation notation these become

$$
\begin{aligned}
Te^x &= \sum_{k=0}^{\infty} \frac{1}{k!}\, x^k , \\
T\sin(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} , \\
T\cos(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k} , \\
T\sinh(x) &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\, x^{2k+1} , \\
T\cosh(x) &= \sum_{k=0}^{\infty} \frac{1}{(2k)!}\, x^{2k} , \\
T\ln(1+x) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\, x^k , \\
T(1+x)^p &= 1 + \sum_{k=1}^{\infty} \frac{p(p-1)\cdots(p-k+1)}{k!}\, x^k .
\end{aligned}
\tag{3.2}
$$

Each of these very important expansions should be memorized. This is simplified somewhat by the fact that those for $\sin(x)$, $\cos(x)$, $\sinh(x)$ and $\cosh(x)$ all derive from the one for e^x. Specifically, the ones for $\sinh(x)$ and $\cosh(x)$ are comprised of the odd and even terms respectively of the one for e^x, while the ones for $\sin(x)$ and $\cos(x)$ are identical to those for $\sinh(x)$ and $\cosh(x)$ respectively except for the introduction of an alternating sign. Given these simple relations, one is left with memorizing only three expansions — the ones for e^x, $\ln(1+x)$ and $(1+x)^p$. At least you should be able to crank out the first several terms without too much trouble.

The idea of this section is simple — just as the derivative of any elementary function can be built up from combinations of a few basic ones, the general Taylor expansion of any elementary function can be built up from combinations of those given in (3.1) or (3.2). Mastering these techniques makes the job of computing Taylor expansions far easier than generating them each time from (2.3) or (2.4).

3.2: Using Linearity. Because taking derivatives is a linear operation, given any two functions f and g that are infinitely differentiable at c and any constant a, the general Taylor expansion of af and $f+g$ at c are easily computed from those of f and g because

$$T_c[af(x)] = a\,T_cf(x)\,, \qquad T_c[f(x)+g(x)] = T_cf(x) + T_cg(x)\,. \tag{3.3}$$

When we add Taylor approximations of the same order we just add the coefficients of like powers of $(x-c)^k$. When we add Taylor approximations of the different order, we first reduce the one of higher order to the lower order before adding the coefficients of like powers of $(x-c)^k$.

Example: To compute $Tf(x)$ for $f(x) = e^x + \sin(x)$ one simply combines the first and thirst expansions in (3.1) obtain

$$\begin{aligned} Tf(x) &= T\exp(x) + T\cos(x) \\ &= \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots\right) + \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots\right) \\ &= 2 + x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \cdots\,. \end{aligned}$$

Notice that the x^2 terms have dropped out. What will be the next power of x whose terms will drop out?

Example: In a similar spirit, to compute $Tg(x)$ for $g(x) = e^x - \cos(x)$, one obtains

$$\begin{aligned} Tg(x) &= T\exp(x) - T\cos(x) \\ &= \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots\right) - \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots\right) \\ &= x + x^2 + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots . \end{aligned}$$

Notice that now the x^0 and x^4 terms have dropped out. What will be the next power of x whose terms will drop out?

3.3: Using Multiplication by and Substitution of Powers. Suppose you know the Taylor expansion of a function f at c:

$$T_c f(x) = f(c) + f'(c)\,(x-c) + \frac{1}{2}f''(c)\,(x-c)^2 + \cdots + \frac{1}{n!}f^{(n)}(c)\,(x-c)^n + \cdots . \tag{3.4}$$

Suppose the function g is given in terms of f by

$$g(x) = (x-c)^m f(x)\,,$$

for some positive integer m. In other words, by the multiplication of f by $(x-c)^m$. Then you can easily compute the Taylor expansion of g at c by simply multiplying each term in (3.4) by $a(x-c)^m$. Thus, you find

$$\begin{gathered} T_c g(x) = f(c)\,(x-c)^m + f'(c)\,(x-c)^{m+1} + \frac{1}{2}f''(c)\,(x-c)^{m+2} + \cdots \\ \cdots + \frac{1}{n!}f^{(n)}(c)\,(x-c)^{m+n} + \cdots\,, \end{gathered} \tag{3.5}$$

which in the summation notation looks like

$$T_c g(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(c)\,(x-c)^{m+k}\,. \tag{3.6}$$

Example: To expand $x^3\sinh(x)$ at 0 one reads off from (3.1) that

$$T[x^3\sinh(x)] = x^4 + \frac{1}{6}x^6 + \frac{1}{120}x^8 + \cdots + \frac{1}{(2n+1)!}x^{2n+4} + \cdots .$$

When this is expressed in summation notation of (3.2) you find

$$T[x^3 \sinh(x)] = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+4} .$$

Suppose you know the Taylor expansion of a function f at 0:

$$Tf(z) = f(0) + f'(0)\, z + \frac{1}{2} f''(0)\, z^2 + \cdots + \frac{1}{n!} f^{(n)}(0)\, z^n + \cdots . \tag{3.7}$$

For example, f could be one of the elementary functions whose Taylor expansion is given in (3.1). Suppose the function g is given in terms of f by

$$g(x) = f\big(a(x-c)^m\big) ,$$

for some constants a and c, and some positive integer m. In other words, by the substitution of z with $a(x-c)^m$ in $f(z)$. Then you can easily compute the Taylor expansion of g at c by simply replacing z by $a(x-c)^m$ everywhere in (3.7). Thus, you find

$$\begin{aligned} T_c g(x) = f(0) + f'(0)\, a\, (x-c)^m + \frac{1}{2} f''(0)\, a^2 (x-c)^{2m} + \cdots \\ \cdots + \frac{1}{n!} f^{(n)}(0)\, a^n (x-c)^{nm} + \cdots , \end{aligned} \tag{3.8}$$

which in the summation notation looks like

$$T_c g(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0)\, a^k (x-c)^{km} . \tag{3.9}$$

Example: To expand $\sin(x^3)$ at 0 one reads off from (3.1) that

$$T \sin(x^3) = x^3 - \frac{1}{6} x^9 + \frac{1}{120} x^{15} - \cdots + \frac{(-1)^n}{(2n+1)!} x^{6n+3} + \cdots .$$

When this is expressed in summation notation of (3.2) you find

$$T \sin(x^3) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{6k+3} .$$

Example: To compute $T[x^3 \cosh(x^2)]$ one reads off from (3.1) that

$$\begin{aligned} x^3 \cosh(x^2) &= x^3 \left(1 + \frac{1}{2}x^4 + \frac{1}{24}x^8 + \frac{1}{720}x^{12} + \cdots \right) \\ &= x^3 + \frac{1}{2}x^7 + \frac{1}{24}x^{11} + \frac{1}{720}x^{15} + \cdots . \end{aligned}$$

Example: To compute Te^{-x} one reads off from (3.1) that

$$Te^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{720}x^6 - \frac{1}{5040}x^7 + \cdots .$$

When this is expressed in summation notation you find

$$Te^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k .$$

Remark: Recall that the definitions of sinh and cosh are

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

The Taylor expansions for sinh and cosh given in (3.1) can then be seen to be the appropriate linear combinations of those of e^x and e^{-x}.

Remark: Using these techniques you can show that even functions have Taylor expansions with only even powers of x while odd functions have expansions with only odd powers.

3.4: Using Identities. Notice that all the Taylor expansions of the elementary functions given in (3.1) and (3.2) are based at 0. This is because one may use identities to reduce the calculation of the Taylor expansion at any point c to that of expansions at 0. For example, if we let $z = x - c$, so that $x = c + z$, then one has the addition formulas

$$\begin{aligned} e^x &= e^{c+z} = e^c e^z , \\ \sin(x) &= \sin(c+z) = \cos(c)\sin(z) + \sin(c)\cos(z) , \\ \cos(x) &= \cos(c+z) = \cos(c)\cos(z) - \sin(c)\sin(z) , \\ \sinh(x) &= \sinh(c+z) = \cosh(c)\sinh(z) + \sinh(c)\cosh(z) , \\ \cosh(x) &= \cosh(c+z) = \cosh(c)\cosh(z) + \sinh(c)\sinh(z) , \end{aligned} \tag{3.10}$$

Hence, by linearity (3.3) and substitution (3.8) the calculation is reduced to knowing five of the expansions listed in (3.2). For example, from the first and third addition formulas in (3.8) we see that

$$\begin{aligned} T_c e^x &= \sum_{k=0}^{\infty} \frac{e^c}{k!}(x-c)^k \,, \\ T_c \cos(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k \cos(c)}{(2k)!}(x-c)^{2k} - \sum_{k=0}^{\infty} \frac{(-1)^k \sin(c)}{(2k+1)!}(x-c)^{2k+1} \\ &= \sum_{k=0}^{\infty} \left[\frac{(-1)^k \cos(c)}{(2k)!}(x-c)^{2k} - \frac{(-1)^k \sin(c)}{(2k+1)!}(x-c)^{2k+1} \right]. \end{aligned}$$

Example: To find $T_{\frac{\pi}{6}} \cos(x)$ one has

$$T_{\frac{\pi}{6}} \cos(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}(x - \tfrac{\pi}{6}) - \frac{\sqrt{3}}{4}(x - \tfrac{\pi}{6})^2 + \frac{1}{12}(x - \tfrac{\pi}{6})^3 + \cdots .$$

Similarly, for $c > 0$ we have the identities

$$\begin{aligned} \ln(x) &= \ln(c+z) = \ln(c) + \ln\left(1 + \frac{z}{c}\right), \\ x^p &= (c+z)^p = c^p \left(1 + \frac{z}{c}\right)^p . \end{aligned} \tag{3.11}$$

Hence, the calculation is reduced to knowing remaining two expansions listed in (3.2):

$$\begin{aligned} T_c \ln(x) &= \ln(c) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{x-c}{c}\right)^k , \\ T_c x^p &= c^p + \sum_{k=1}^{n} \frac{p(p-1)\cdots(p-k+1)}{k!} \, c^p \left(\frac{x-c}{c}\right)^k . \end{aligned}$$

Notice that this is a far quicker way to derive (2.22) and (2.24) than using (2.4).

Example: To find $T_3 \ln(x)$ one has

$$T_3 \ln(x) = \ln(3) + \frac{1}{3}(x-3) - \frac{1}{18}(x-3)^2 + \frac{1}{81}(x-3)^3 - \frac{1}{324}(x-3)^4 + \cdots .$$

4. LIMITS WITH INDETERMINANT FORMS

4.1: Taylor Approximations Applied to Limits. Taylor approximations can help you evaluate limits of the form

$$\lim_{x\to c} \frac{f(x)}{g(x)}\,,$$

in cases where $f(c) = g(c) = 0$. Such a limit is said to have the indeterminant form 0/0.

The key idea is that the behavior of a function f near a point c is governed by the first nonzero term in its Taylor expansion at c in the sense that their ratio approaches one. More precisely, if the first nonzero term in the Taylor expansion occurs at order m then one can show that

$$\lim_{x\to c} \frac{f(x)}{\frac{1}{m!}\,f^{(m)}(c)\,(x-c)^m} = 1\,.$$

We then say that the **leading behavior** of f near c is given by

$$f(x) \sim \frac{1}{m!} f^{(m)}(c)(x-c)^m\,.$$

If f and g have Taylor expansions at c whose first nonzero terms occur at orders m and n respectively, then their leading behaviors near c are given by

$$f(x) \sim \frac{1}{m!} f^{(m)}(c)(x-c)^m\,, \qquad\qquad g(x) \sim \frac{1}{n!} g^{(n)}(c)(x-c)^n\,,$$

where both $f^{(m)}(c)$ and $g^{(n)}(c)$ are nonzero. The leading behavior of f/g is then given by

$$\frac{f(x)}{g(x)} \sim \frac{n!}{m!}\,\frac{f^{(m)}(c)}{g^{(n)}(c)}\,(x-c)^{m-n}\,. \tag{4.1}$$

We can read off from this that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \begin{cases} 0 & \text{if } m > n\,, \\ \dfrac{f^{(n)}(c)}{g^{(n)}(c)} & \text{if } m = n\,, \\ \text{diverges} & \text{if } m < n\,. \end{cases} \tag{4.2}$$

In fact, more refined information can be read off from (4.1) about the nature of the divergence when $m < n$. In that case if we let $\pm$ correspond to the sign of $f^{(m)}(c)/g^{(n)}(c)$, then when $n - m$ is even

$$\lim_{x\to c} \frac{f(x)}{g(x)} \text{ diverges to } \pm\infty\,, \tag{4.3}$$

while when $n-m$ is odd

$$\lim_{x\to c^+}\frac{f(x)}{g(x)} \text{ diverges to } \pm\infty\,, \qquad \lim_{x\to c^-}\frac{f(x)}{g(x)} \text{ diverges to } \mp\infty\,. \tag{4.4}$$

The above analysis was devised by l'Hopital to evaluate limits that have the indeterminant form 0/0.

In order to apply (4.2), you only need to find the first n at which at least one of either $f^{(n)}(c)$ or $g^{(n)}(c)$ does not vanish. Then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \begin{cases} \dfrac{f^{(n)}(c)}{g^{(n)}(c)} & \text{if } g^{(n)}(c)\neq 0\,, \\ \text{diverges} & \text{if } g^{(n)}(c)=0\,. \end{cases} \tag{4.5}$$

Example: To evaluate

$$\lim_{h\to 0}\frac{e^h-1-h}{\sin^2(h)}\,,$$

use the Taylor expansions

$$Te^h = 1+h+\frac{1}{2}h^2+\cdots\,, \qquad T\sin(h) = h-\frac{1}{6}h^3+\cdots\,,$$

to obtain that the leading behaviors of the numerator and denominator are given by

$$e^h-1-h\sim\frac{1}{2}h^2\,, \qquad \sin^2(h)\sim h^2\,.$$

Hence,

$$\lim_{h\to 0}\frac{e^h-1-h}{\sin^2(h)} = \lim_{h\to 0}\frac{\frac{1}{2}h^2}{h^2} = \frac{1}{2}\,,$$

where we have divided the numerator and denominator by h^2 before taking the limit in the last step.

Example: To evaluate

$$\lim_{x\to 0}\frac{\sin(x)-x}{\sin(x^3)}\,,$$

use the Taylor expansions

$$T\sin(x) = x-\frac{1}{6}x^3+\cdots\,, \qquad T\sin(x^3) = x^3+\cdots\,,$$

to obtain that the leading behaviors of the numerator and denominator are given by

$$\sin(x) - x \sim -\frac{1}{6}x^3 \,, \qquad \sin(x^3) \sim x^3 \,.$$

Hence,

$$\lim_{x\to 0} \frac{\sin(x) - x}{\sin^3(x)} = \lim_{x\to 0} \frac{-\frac{1}{6}x^3}{x^3} = -\frac{1}{6} \,,$$

where we have divided the numerator and denominator by x^3 before taking the limit in the last step.

4.2: Generalized l'Hopital Rule. Derivatives can also help you evaluate limits of the indeterminant form 0/0 through the following.

Generalized l'Hopital Rule: Let f and g be differentiable over an interval (a,b) with $g'(x) \neq 0$ for every x in (a,b). Suppose that either

$$\lim f(x) = 0 \,, \qquad \text{and} \qquad \lim g(x) = 0 \,, \tag{4.6}$$

or

$$\lim f(x) = \pm\infty \,, \qquad \text{and} \qquad \lim g(x) = \pm\infty \,, \tag{4.7}$$

where lim stands for either

$$\lim_{x\to a^+} \qquad \text{or} \qquad \lim_{x\to b^-} \,.$$

Then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} \,, \tag{4.8}$$

whenever the limit on the right-hand side exists.

Example: To evaluate

$$\lim_{x\to 0} \frac{\tan(x^2)}{\sin^2(x)} \,,$$

you can use the generalized l'Hopital rule twice to obtain

$$\begin{aligned} \lim_{x\to 0} \frac{\tan(x^2)}{\sin^2(x)} &= \lim_{x\to 0} \frac{\sec^2(x^2)2x}{2\sin(x)\cos(x)} && \left[\frac{0}{0}\right] \\ &= \lim_{x\to 0} \frac{\sec^2(x^2)2 + 8\sec^2(x^2)\tan(x^2)x^2}{2\cos^2(x) - 2\sin^2(x)} && = \frac{2}{2} = 1 \,. \end{aligned}$$

Example: To evaluate

$$\lim_{x\to 0} \frac{\sin(x) - x}{\sin(x^3)}\,,$$

you can use the generalized l'Hopital rule three times to obtain

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x) - x}{\sin(x^3)} &= \lim_{x\to 0} \frac{\cos(x) - 1}{\cos(x^3)3x^2} & \left[\frac{0}{0}\right] \\
&= \lim_{x\to 0} \frac{-\sin(x)}{\cos(x^3)6x - \sin(x^3)9x^4} & \left[\frac{0}{0}\right] \\
&= \lim_{x\to 0} \frac{-\cos(x)}{\cos(x^3)6 - \sin(x^3)54x^3 - \cos(x^3)9x^4} \quad = \frac{-1}{6} = -\frac{1}{6}\,.
\end{aligned}$$

This calulation should be contrasted with the Taylor approximation approach found at the bottom of page 19.

The generalized l'Hopital rule can also be applied when

- $\lim f(x) = \pm\infty$ and $\lim g(x) = \pm\infty$,
- $a = \pm\infty$.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}\,,$$

whenever the limit on the right-hand side exists.

Example: To evaluate

$$\lim_{x\to\infty} \frac{\tanh(x^2)}{\sinh^2(x)}\,,$$

you can use the generalized l'Hopital rule twice to obtain

$$\begin{aligned}
\lim_{x\to\infty} \frac{\tanh(x^2)}{\sinh^2(x)} &= \lim_{x\to\infty} \frac{\operatorname{sech}^2(x^2)2x}{2\sinh(x)\cosh(x)} & \left[\frac{\infty}{\infty}\right] \\
&= \lim_{x\to\infty} \frac{\operatorname{sech}^2(x^2)2 - 8\operatorname{sech}^2(x^2)\tanh(x^2)x^2}{2\cosh^2(x) + 2\sinh^2(x)} \quad = \frac{2}{\infty} = 0\,.
\end{aligned}$$