First In-Class Exam Solutions: Math 410
Section 0501, Professor Levermore
Friday, 1 October 2010

1. [10] Let X be a field. Use the field axioms to show that if $x, y \in X$ such that $xy = 1$ then $y = x^{-1}$.

   Remark. The main point to keep in mind when doing problems like this is *to justify every step in your solution either by one or more of the axioms or by a previous step.*

   Solution. First $x \neq 0$ because if $x = 0$ then $1 = xy = 0y = 0$, which is a contradiction. (The fact that $0y = 0$ for every $y \in X$ was shown in class.) Then

$$
\begin{aligned}
y &= y\,1 && \text{(mult. ident. axiom)} \\
&= 1\,y && \text{(mult. comm. axiom)} \\
&= \big(xx^{-1}\big)\,y && \text{(mult. inv. axiom and } x \neq 0) \\
&= \big(x^{-1}x\big)\,y && \text{(mult. comm. axiom)} \\
&= x^{-1}(xy) && \text{(mult. assoc. axiom)} \\
&= x^{-1}1 && \text{(because } xy = 1) \\
&= x^{-1} && \text{(mult. ident. axiom)}\,.
\end{aligned}
$$

   Remark. For completeness, here is the proof that $0y = 0$. Let $y \in X$. The additive identity axiom implies $0 = 0 + 0$. The distributive axiom then gives the equality

$$
y0 = y(0+0) = y0 + y0\,.
$$

   Then

$$
\begin{aligned}
0 &= y0 + \big(-(y0)\big) && \text{(add. inv. axiom)} \\
&= (y0 + y0) + \big(-(y0)\big) && \text{(above equality)} \\
&= y0 + \big(y0 + \big(-(y0)\big)\big) && \text{(add. assoc. axiom)} \\
&= y0 + 0 && \text{(add. inv. axiom)} \\
&= y0 && \text{(add. indent. axiom)} \\
&= 0y && \text{(add. comm. axiom)}\,.
\end{aligned}
$$

2. [15] Give a counterexample to each of the following false assertions.
   (a) If a sequence $\{a_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ is divergent then the subsequence $\{a_{2k}\}_{k\in\mathbb{N}}$ is divergent.
   (b) A countable intersection of nested nonempty open intervals is also nonempty.
   (c) If $\lim_{k\to\infty} a_k = 0$ then $\displaystyle\sum_{k=0}^{\infty} a_k$ converges.

Solution (a). A simple counterexample is obtained by setting $a_k = (-1)^k$ because

$$\{(-1)^k\} \quad \text{diverges}\,, \quad \text{while} \quad \lim_{k\to\infty} (-1)^{2k} = 1\,.$$

Solution (b). A countable intersection of nested nonempty open intervals must have the form

$$\bigcap_{k=0}^{\infty} (a_k, b_k)$$

where $a_k < b_k$ and $(a_{k+1}, b_{k+1}) \subset (a_k, b_k)$ for every $k \in \mathbb{N}$. Such an intersection that is empty is obtained by setting $a_k = 0$ and $b_k = 2^{-k}$ for every $k \in \mathbb{N}$.

Solution (c). A simple counterexample is obtained by setting $a_k = 1/(k+1)$ because

$$\lim_{k\to\infty} \frac{1}{k+1} = 0\,, \quad \text{while the harmonic series} \quad \sum_{k=0}^{\infty} \frac{1}{k+1} \quad \text{diverges}\,.$$

3. [15] Consider the real sequence $\{c_k\}_{k\in\mathbb{N}}$ given by

$$c_k = (-1)^k \, \frac{k+3}{2k+2} \qquad \text{for every } k \in \mathbb{N}\,,$$

with the convention that $\mathbb{N} = \{0, 1, 2, \dots\}$.
   (a) Write down the first three terms of the subsequence $\{c_{2k+1}\}_{k\in\mathbb{N}}$.
   (b) Write down $\liminf_{k\to\infty} c_k$ and $\limsup_{k\to\infty} c_k$. (No proof is needed here.)

Solution (a). You are given that $\mathbb{N} = \{0, 1, 2, \cdots\}$, as was the convention in class and in the notes (but in not the book). The first three terms of the subsequence $\{c_{2k+1}\}_{k\in\mathbb{N}}$ are therefore

$$c_1 = -\tfrac{4}{4} = -1\,, \qquad c_3 = -\tfrac{6}{8} = -\tfrac{3}{4}\,, \qquad c_5 = -\tfrac{8}{12} = -\tfrac{2}{3}\,.$$

Solution (b). Because $c_{2k} > 0$ while $c_{2k+1} < 0$, and because

$$\lim_{k\to\infty} c_{2k} = \lim_{k\to\infty} \frac{2k+3}{4k+2} = \frac{1}{2}\,,$$

while

$$\lim_{k\to\infty} c_{2k+1} = -\lim_{k\to\infty} \frac{2k+4}{4k+4} = -\frac{1}{2}\,,$$

one has that

$$\limsup_{k\to\infty} c_k = \frac{1}{2}\,, \qquad \liminf_{k\to\infty} c_k = -\frac{1}{2}\,.$$

4. [15] Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be bounded, positive sequences in $\mathbb{R}$.

(a) Prove that

$$\limsup_{k\to\infty}\,(a_k b_k) \leq \left(\limsup_{k\to\infty}\, a_k\right)\left(\limsup_{k\to\infty}\, b_k\right).$$

(b) Write down an example for which equality does not hold above.

Solution (a). Let $c_k = a_k b_k$ for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ define

$$\overline{a}_k = \sup\{a_l \,:\, l \geq k\}\,, \qquad \overline{b}_k = \sup\{b_l \,:\, l \geq k\}\,, \qquad \overline{c}_k = \sup\{c_l \,:\, l \geq k\}\,.$$

Because the sequences $\{a_k\}$, $\{b_k\}$, and $\{c_k\}$ are positive and bounded above, the sequences $\{\overline{a}_k\}$, $\{\overline{b}_k\}$, and $\{\overline{c}_k\}$, are positive and nonincreasing. The Monotonic Sequence Theorem thereby implies that the sequences $\{\overline{a}_k\}$, $\{\overline{b}_k\}$, and $\{\overline{c}_k\}$ are convergent. By the definition of lim sup we then have

$$\limsup_{k\to\infty} a_k = \lim_{k\to\infty}\,\overline{a}_k\,, \qquad \limsup_{k\to\infty} b_k = \lim_{k\to\infty}\,\overline{b}_k\,, \qquad \limsup_{k\to\infty} c_k = \lim_{k\to\infty}\,\overline{c}_k\,.$$

The key observation is that for every $k \in \mathbb{N}$ we have

$$c_l = a_l b_l \leq \overline{a}_k \overline{b}_k \quad \text{for every } l \geq k\,,$$

which yields the inequality

$$\overline{c}_k = \sup\{c_l \,:\, l \geq k\} \leq \overline{a}_k \overline{b}_k\,.$$

This inequality and the properties of limits then implies

$$\begin{aligned}
\limsup_{k\to\infty}\,(a_k b_k) &= \limsup_{k\to\infty}\, c_k \\
&= \lim_{k\to\infty}\, \overline{c}_k \\
&\leq \lim_{k\to\infty}\, (\overline{a}_k \overline{b}_k) \\
&= \left(\lim_{k\to\infty}\, \overline{a}_k\right)\left(\lim_{k\to\infty}\, \overline{b}_k\right) \\
&= \left(\limsup_{k\to\infty}\, a_k\right)\left(\limsup_{k\to\infty}\, b_k\right).
\end{aligned}$$

Solution (b). Let $a_k = 2^{(-1)^k}$ and $b_k = 2^{(-1)^{k+1}}$ for every $k \in \mathbb{N}$. Clearly

$$\limsup_{k\to\infty} a_k = \lim_{k\to\infty}\, a_{2k} = 2\,, \qquad \limsup_{k\to\infty} b_k = \lim_{k\to\infty}\, b_{2k+1} = 2\,,$$

while (because $a_k b_k = 1$ for every $k \in \mathbb{N}$)

$$\limsup_{k\to\infty}\,(a_k b_k) = \lim_{k\to\infty}\,(a_k b_k) = 1\,.$$

Therefore

$$\limsup_{k\to\infty}\,(a_k b_k) = 1 < 4 = 2\cdot 2 = \left(\limsup_{k\to\infty}\, a_k\right)\left(\limsup_{k\to\infty}\, b_k\right).$$

5. [10] Let X^c denote the closure of any subset X of $\mathbb{R}$. Let A and B be subsets of $\mathbb{R}$. Prove that $A^c \cup B^c \subset (A \cup B)^c$.

   Remark. You must show that every element of $A^c \cup B^c$ is also an element of $(A \cup B)^c$. If your proof directly uses the definition of closure then its first step should be clear.

   Solution. Let $x \in A^c \cup B^c$ be arbitrary. Then either $x \in A^c$ or $x \in B^c$. (Both can be true.) Without loss of generality we can assume that $x \in A^c$. By the definition of closure, there exists a sequence $\{x_k\}_{k \in \mathbb{N}}$ contained in A such that $x_k \to x$ as $k \to \infty$. But the sequence $\{x_k\}_{k \in \mathbb{N}}$ is therefore contained in both $A \cup B$ while $x_k \to x$ as $k \to \infty$. By the definition of closure, it follows that $x \in (A \cup B)^c$. But because $x \in A^c \cup B^c$ was arbitrary, we conclude that $A^c \cup B^c \subset (A \cup B)^c$.

   Remark. You could also have built a proof around the fact that if $C \subset D$ then their closures satisfy $C^c \subset D^c$. (This is a fact you should be able to prove directly from the definition of closure.)

   Alternative Solution. Because $A \subset (A \cup B)$ and $B \subset (A \cup B)$, we know $A^c \subset (A \cup B)^c$ and $B^c \subset (A \cup B)^c$. We conclude that $A^c \cup B^c \subset (A \cup B)^c$.

6. [15] Determine all $a \in \mathbb{R}$ for which

$$\sum_{k=0}^{\infty} \left(\frac{k^2+1}{k^4+1} \right)^a \quad \text{converges}\,.$$

   Give your reasoning.

   Solution. The series converges for $a \in (\frac{1}{2}, \infty)$ and diverges otherwise. Because

$$\frac{k^2+1}{k^4+1} \sim \frac{1}{k^2} \quad \text{as } k \to \infty\,,$$

   one sees that the original series should be compared with the p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^{2a}}\,.$$

   This is best handled by Limit Comparison Test. Indeed, because for every $a \in \mathbb{R}$ one has

$$\lim_{k \to \infty} \frac{\left(\dfrac{k^2+1}{k^4+1} \right)^a}{\dfrac{1}{k^{2a}}} = \lim_{k \to \infty} \left(\frac{k^4+k^2}{k^4+1} \right)^a = 1\,,$$

   the Limit Comparison Test then implies that

$$\sum_{k=0}^{\infty} \left(\frac{k^2+1}{k^4+1} \right)^a \quad \text{converges} \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} \frac{1}{k^{2a}} \quad \text{converges}\,.$$

   Because the $p = 2a$ for the p-series, it converges for $a \in (\frac{1}{2}, \infty)$ and diverges otherwise. The same is therefore true for the original series.

7. [10] Let $\{a_k\}_{k\in\mathbb{N}}$ be a real sequence and $\{a_{n_k}\}$ be any subsequence of it. Show that

$$\sum_{k=0}^{\infty} a_k \quad \text{converges absolutely} \qquad \Longrightarrow \qquad \sum_{k=0}^{\infty} a_{n_k} \quad \text{converges absolutely}\,.$$

Solution. By the definition of absolute convergence

$$\sum_{k=0}^{\infty} a_k \quad \text{converges absolutely} \qquad \Longleftrightarrow \qquad \sum_{k=0}^{\infty} |a_k| \quad \text{converges}\,,$$

$$\sum_{k=0}^{\infty} a_{n_k} \quad \text{converges absolutely} \qquad \Longleftrightarrow \qquad \sum_{k=0}^{\infty} |a_{n_k}| \quad \text{converges}\,.$$

For every $m, n \in \mathbb{N}$ define the sequences $\{p_m\}$ and $\{q_n\}$ of partial sums

$$p_m = \sum_{k=0}^{m} |a_{n_k}|\,, \qquad q_n = \sum_{k=0}^{n} |a_k|\,.$$

It is clear that these sequences are nondecreasing and that the Monotonic Sequence Theorem implies

$$\sum_{k=0}^{\infty} |a_k| \quad \text{converges} \qquad \Longleftrightarrow \qquad \{q_n\} \text{ is bounded above}\,,$$

$$\sum_{k=0}^{\infty} |a_{n_k}| \quad \text{converges} \qquad \Longleftrightarrow \qquad \{p_m\} \text{ is bounded above}\,.$$

Moreover p_m and q_n satisfy the inequality

$$p_m = \sum_{k=0}^{m} |a_{n_k}| \le \sum_{k=0}^{n_m} |a_k| = q_{n_m} \quad \text{for every } m \in \mathbb{N}\,.$$

This inquality shows that if $\{q_n\}$ is bounded above then $\{p_m\}$ is bounded above. Hence,

$$\begin{aligned}
\sum_{k=0}^{\infty} a_k \quad \text{converges absolutely} \qquad &\Longleftrightarrow \qquad \{q_n\} \text{ is bounded above} \\
&\Longrightarrow \qquad \{p_m\} \text{ is bounded above} \\
&\Longleftrightarrow \qquad \sum_{k=0}^{\infty} a_{n_k} \quad \text{converges absolutely}\,.
\end{aligned}$$

8. [10] Let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let A be a subset of $\mathbb{R}$. Write the negations of the following assertions.
   (a) "There exists $m \in \mathbb{R}$ such that $b_j > m$ eventually as $j \to \infty$."
   (b) "Every sequence in A has a subsequence that converges to a limit in A."

   Solution (a). "For every $m \in \mathbb{R}$ one has $b_j \le m$ frequently as $j \to \infty$."

   Solution (b). "There is a sequence in A such that every subsequence of it either diverges or converges to a limit outside A."

   Remark. The answer "There is a sequence in A such that no subsequence of it converges to a limit in A." does not fully carry the negation through.