Second In-Class Exam Solutions: Math 410
Section 0501, Professor Levermore
Friday, 5 November 2010

1. [10] Give a counterexample to each of the following false assertions.
   (a) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and increasing over $\mathbb{R}$ then $f' > 0$ over $\mathbb{R}$.

   Solution. A simple example that we discussed in class is given by
$$f(x) = x^3 \quad \text{for every } x \in \mathbb{R}\,.$$
   Clearly $f : \mathbb{R} \to \mathbb{R}$ is differentiable and increasing over $\mathbb{R}$, yet $f'(x) = 3x^2$ vanishes at $x = 0$.

   (b) If $f : (a, b) \to \mathbb{R}$ is continuous then f has a minimum or a maximum over (a, b).

   Solution. A simple example that we discussed in class is given by
$$f(x) = x \quad \text{for every } x \in (-1, 1)\,.$$
   Clearly $f : (-1, 1) \to \mathbb{R}$ is continuous, yet does not have a minimum or a maximum over $(-1, 1)$.

2. [10] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove it is continuous.

   Solution. Let $a \in \mathbb{R}$ be arbitrary. Because f is differentialble at a one knows that
$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)\,.$$
   Because for every $x \neq a$ one has the identity
$$f(x) = f(a) + \frac{f(x) - f(a)}{x - a}\,(x - a)\,,$$
   by the proporties of limits we see that
$$\begin{aligned}\lim_{x \to a} f(x) &= \lim_{x \to a} \left(f(a) + \frac{f(x) - f(a)}{x - a}\,(x - a) \right) \\ &= \lim_{x \to a} f(a) + \lim_{x \to a} \frac{f(x) - f(a)}{x - a}\, \lim_{x \to a} (x - a) \\ &= f(a) + f'(a) \cdot 0 = f(a)\,.\end{aligned}$$
   Hence, f is continuous at a. But $a \in \mathbb{R}$ was arbitrary, so therefore $f : \mathbb{R} \to \mathbb{R}$ is continuous.

3. [15] Show that
$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} \quad \text{for every } x \in \mathbb{R}\,.$$

   Solution. Let $f(x) = \sin(x)$. Then for every $k \in \mathbb{N}$ one has
$$f^{(2k)}(x) = (-1)^k \sin(x)\,, \qquad f^{(2k+1)}(x) = (-1)^k \cos(x)\,.$$
   Because
$$f^{(2k)}(0) = 0\,, \qquad f^{(2k+1)}(0) = (-1)^k\,,$$

the series is just the formal Taylor series for f centered at 0. Moreover, we see that the n^{th} partial sum can be expressed as a Taylor polynomial approximation in two ways:

$$\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} = T_0^{(2n+1)} \sin(x) = T_0^{(2n+2)} \sin(x)\,.$$

If we use the last expression then the Lagrange Remainder Theorem states that for every $x \in \mathbb{R}$ there exists some p between 0 and x such that

$$\sin(x) = T_0^{(2n+2)} \sin(x) + \frac{(-1)^{n+1}}{(2n+3)!} \cos(p) x^{2n+3}\,.$$

Hence, for every $x \in \mathbb{R}$

$$\left| \sin(x) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} \right| \leq \frac{1}{(2n+3)!} |x|^{2n+3}\,.$$

Because for every $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{(2n+3)!} |x|^{2n+3} = 0\,,$$

the sequence of partial sums therefore converges to $\sin(x)$. □

4. [15] Prove that for every $x > -1$ one has

$$1 + \tfrac{5}{4}x \leq (1+x)^{\frac{5}{4}}\,.$$

Solution. The most direct approach to this problem uses the Lagrange Remainder Theorem. Define $f(x) = (1+x)^{\frac{5}{4}}$ for every $x > -1$. Then f is twice differentiable with

$$f'(x) = \tfrac{5}{4}(1+x)^{\frac{1}{4}}\,, \qquad f''(x) = \tfrac{5}{16}(1+x)^{-\frac{3}{4}}\,.$$

By the Lagrange Remainder Theorem for every $x > -1$ there exists a p between 0 and x such that

$$f(x) = f(0) + f'(0)x + \tfrac{1}{2} f''(p) x^2\,.$$

Hence,

$$(1+x)^{\frac{1}{4}} - 1 - \tfrac{5}{4}x = \tfrac{5}{32}(1+p)^{-\frac{3}{4}} x^2 \geq 0\,.$$

The result follows. □

Second Solution. Another approach to this problem uses the Monotonicity Theorem. Define $g(x) = (1+x)^{\frac{5}{4}} - 1 - \frac{5}{4}x$ for every $x > -1$. Then g is differentiable with

$$g'(x) = \tfrac{5}{4}\left[(1+x)^{\frac{1}{4}} - 1\right].$$

Clearly, $g'(x) < 0$ for $x \in (-1, 0)$ while $g'(x) > 0$ for $x \in (0, \infty)$. By the Monotonicity Theorem, g is decreasing over $x \in (-1, 0]$ and g is increasing over $[0, \infty)$. Therefore 0 is a global minimizer of g over $(-1, \infty)$, and $g(0) = 0$ is the minimum of g over $(-1, \infty)$. Hence, for every $x > -1$ we have

$$(1+x)^{\frac{5}{4}} - 1 - \tfrac{5}{4}x = g(x) \geq g(0) = 0\,.$$

The result follows. □

5. [10] Evaluate the following limit. Give your reasoning. (You may use theorems we have proved in class.)

$$\lim_{x\to 3} \frac{x^4 - 81}{x^2 - 9}\,.$$

Solution. For every $x \neq 1$ one has

$$\frac{x^4 - 81}{x^2 - 9} = x^2 + 9\,.$$

Because the right-hand side above is continuous over $\mathbb{R}$, one has

$$\lim_{x\to 3} \frac{x^4 - 81}{x^2 - 9} = \lim_{x\to 3} x^2 + 9 = 9 + 9 = 18\,.$$

Second Solution. Because the limit has a 0/0 indeterminant form, by the l'Hopital rule and the continuity of rational functions we obtain

$$\lim_{x\to 3} \frac{x^4 - 81}{x^2 - 9} = \lim_{x\to 3} \frac{4x^3}{2x} = \frac{4\cdot 3^3}{2\cdot 3} = 2\cdot 3^2 = 18\,.$$

6. [15] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose the following equation has at most one solution:

$$f'(x) = 0\,, \qquad x \in \mathbb{R}\,.$$

Show the following equation has at most two solutions:

$$f(x) = 0\,, \qquad x \in \mathbb{R}\,.$$

Solution. Suppose that the equation $f'(x) = 0$ has at most one solution while the equation $f(x) = 0$ has (at least) three solutions $\{x_0, x_1, x_2\}$. Without loss of generality we can assume that

$$-\infty < x_0 < x_1 < x_2 < \infty\,.$$

Then for each $i = 1, 2$ one knows that

- $f : [x_{i-1}, x_i] \to \mathbb{R}$ is differentiable (and hence continuous),
- $f(x_{i-1}) = f(x_i) = 0$.

Rolle's Theorem then implies that for each $i = 1, 2$ there exists a point $p_i \in (x_{i-1}, x_i)$ such that $f'(p_i) = 0$. Because the intervals (x_0, x_1) and (x_1, x_2) are disjoint, the points p_1 and p_2 are distinct. The equation $f'(x) = 0$ therefore has at least two solutions, which contradicts our starting supposition. □

Second Solution. There are two cases to consider: either $f'(x) = 0$ has no solutions or it has exactly one solution.

If $f'(x) = 0$ has no solutions over $\mathbb{R}$ then by the Sign Dichotomy Theorem f' must be either negative or positive over $\mathbb{R}$. The Monotonicity Theorem then implies that f must be monotonic (and hence one-to-one) over $\mathbb{R}$. The equation $f(x) = 0$ can therefore have at most one solution.

If $f'(x) = 0$ has exactly one solution c then by the Sign Dichotomy Theorem f' must be either negative or positive over each of the disjoint intervals

$$(-\infty, c)\,, \quad (c, \infty)\,.$$

The Monotonicity Theorem then implies that f must be monotonic (and hence one-to-one) over each of the two intervals

$$(-\infty, c]\,, \quad [c, \infty)\,.$$

The equation $f(x) = 0$ can therefore have at most one solution in each of these intervals. Because the union of these intervals is $\mathbb{R}$, the equation $f(x) = 0$ can have at most two solutions. □

Remark. The second solution rests upon the Sign Dichotomy Theorem and the Monotonicity Theorem. This machinery is much heavier than that used in the first solution, which rests only upon Rolle's Theorem. Indeed, the proof of the Monotonicity Theorem rests upon the Mean-Value Theorem, the proof of which rests upon Rolle's Theorem.

7. [15] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be uniformly continuous over D. Let $\{x_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence contained in D. Show that $\{f(x_k)\}_{k\in\mathbb{N}}$ is a convergent sequence.

Solution. Let $\epsilon > 0$. Because $f : (a, b) \to \mathbb{R}$ is uniformly continuous over (a, b), there exists a $\delta > 0$ such that for every $x, y \in D$ one has

$$|x - y| < \delta \quad \Longrightarrow \quad \big|f(x) - f(y)\big| < \epsilon\,.$$

Because $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence, there exists an $N \in \mathbb{N}$ such that for every $k, l \in \mathbb{N}$ one has

$$k, l > N \quad \Longrightarrow \quad |x_k - x_l| < \delta\,.$$

Hence, for every $k, l \in \mathbb{N}$ one has

$$\begin{aligned} k, l > N \quad &\Longrightarrow \quad |x_k - x_l| < \delta \\ &\Longrightarrow \quad \big|f(x_k) - f(x_l)\big| < \epsilon\,. \end{aligned}$$

Therefore $\{f(x_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence. By the Cauchy Criterion Theorem we conclude that $\{f(x_k)\}_{k\in\mathbb{N}}$ is a convergent sequence. □

8. [10] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Let c be a limit point of D. Write negations of the following assertions.

(a) "For every sequence $\{x_k\}_{k\in\mathbb{N}} \subset D - \{c\}$ one has

$$\lim_{k\to\infty} |x_k - c| = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} f(x_k) = \infty\,."$$

Solution. There exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset D - \{c\}$ such that

$$\lim_{k\to\infty} |x_k - c| = 0 \quad \text{and} \quad \liminf_{k\to\infty} f(x_k) < \infty\,.$$

(b) "For every $M \in \mathbb{R}$ there exists a $\delta > 0$ such that for every $x \in D$ one has

$$0 < |x - c| < \delta \quad \Longrightarrow \quad f(x) > M\,."$$

Solution. There exists $M \in \mathbb{R}$ such that for every $\delta > 0$ there exists $x \in D$ such that

$$0 < |x - c| < \delta \quad \text{and} \quad f(x) \le M\,.$$