**Fourth Homework: MATH 410**
**Due Wedesday, 29 September 2010**

1. Exercise 1 of Section 9.1 in the text.
2. Exercise 2 of Section 9.1 in the text.
3. Exercise 3 of Section 9.1 in the text.
4. Exercise 4 of Section 9.1 in the text.
5. Consider a formal infinite series of the form
$$\sum_{k=1}^{\infty} kr^k \,,$$
for some $r \in \mathbb{R}$. Find all the values of $r$ for which this series converges and evaluate the sum. (Hint: Find an explicit expression for the partial sums and evaluate the limit. The explicit expression may be derived from the analogous expression for a geometric series.)
6. The proof of Proposition 3.4 in the notes argues that the direct comparison test applies whenever the limit comparison test applies, and that the limit comparison test applies whenever the ratio comparison test applies. Can you find (a) an example where the direct comparison test applies but the limit comparison test fails, and (b) an example where the limit comparison test applies but the ratio comparison test fails?
7. Let $\{a_k\}$ be a nonincreasing, positive sequence. Prove that
$$\sum_{k=1}^{\infty} a_k \quad \text{converges} \quad \Longleftrightarrow \quad \sum_{k=0}^{\infty} 5^k a_{5^k} \quad \text{converges}\,.$$
8. Complete the proof of Proposition 3.7 in the notes by showing that
$$\lim_{k\to\infty} s_{2k} = s \quad \text{and} \quad \lim_{k\to\infty} s_{2k+1} = s \quad \Longrightarrow \quad \lim_{k\to\infty} s_k = s\,.$$
9. Prove Proposition 3.9 in the notes.
10. Let $\{a_k\}_{k\in\mathbb{N}}$ be a real sequence and $\{a_{n_k}\}$ be any subsequence. Show that
$$\sum_{k=0}^{\infty} a_k \quad \text{converges absolutely} \quad \Longrightarrow \quad \sum_{k=0}^{\infty} a_{n_k} \quad \text{converges absolutely}\,.$$
11. Give examples of both a divergent series and a convergent series such that
$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} = 1\,.$$
12. Consider the set
$$\left\{ x \in \mathbb{R} \,:\, \sum_{n=0}^{\infty} \frac{(4n)!}{n!}\,\frac{(2n)!}{(3n)!}\, x^n \quad \text{converges} \right\}.$$
Use the root test to prove that this set is an interval and find its endpoints. You may use the fact that
$$\lim_{k\to\infty} \frac{\sqrt[k]{k!}}{k} = \frac{1}{e}\,.$$