Tenth Homework: MATH 410
Due Wednesday, 10 November 2010

1. Exercise 1 of Section 3.4 in the text.
2. Exercise 2 of Section 3.4 in the text.
3. Exercise 3 of Section 3.4 in the text.
4. Exercise 6 of Section 3.4 in the text.
5. Exercise 7 of Section 3.4 in the text.
6. Exercise 11 of Section 3.4 in the text.
7. Let $f(x) = \sinh(x)$ for every $x \in \mathbb{R}$. Then for every $k \in \mathbb{N}$ and every $x \in \mathbb{R}$ one has
$$f^{(2k)}(x) = \sinh(x)\,, \qquad f^{(2k+1)}(x) = \cosh(x)\,.$$
Show that
$$\sinh(x) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\, x^{2k+1} \quad \text{for every } x \in \mathbb{R}\,.$$
8. Suppose that $f : (a,b) \to \mathbb{R}$ is twice differentiable and that $f'' : (a,b) \to \mathbb{R}$ is bounded over (a,b). Show that there exists an $M \in \mathbb{R}_+$ such that for all points $x, y \in (a,b)$ one has
$$\big|f'(x) - f'(y)\big| \leq M\,|x-y|\,.$$
9. Prove that for every $x > 0$ one has
$$1 + \tfrac{3}{2}x < (1+x)^{\frac{3}{2}} < 1 + \tfrac{3}{2}x + \tfrac{3}{8}x^2\,.$$
10. Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be Hölder continuous of order $\alpha \in (0,1]$ if there exists a $C \in \mathbb{R}_+$ such that for every $x, y \in D$ one has
$$|f(x) - f(y)| \leq C\,|x-y|^{\alpha}\,.$$
Show that if $f : D \to \mathbb{R}$ is Hölder continuous of order α for some $\alpha \in (0,1]$ then it is uniformly continuous over D.
11. Let $\alpha \in (0,1)$. Let $f : [0,\infty) \to \mathbb{R}$ be defined by $f(x) = x^{\alpha}$. Show that f is uniformly continuous over $[0,\infty)$. Hint: Apply the result of the previous problem after showing that
$$\big|x^{\alpha} - y^{\alpha}\big| \leq |x-y|^{\alpha} \quad \text{for every } x, y \in [0,\infty)\,.$$
12. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Write negations of the following assertions.
    (a) "For all sequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ contained in D one has
$$\lim_{k\to\infty} |x_k - y_k| = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} \big|f(x_k) - f(y_k)\big| = 0\,."$$
    (b) "For every $\epsilon > 0$ there exists a $\delta > 0$ such that for all points $x, y \in D$ one has
$$|x-y| < \delta \quad \Longrightarrow \quad \big|f(x) - f(y)\big| < \epsilon\,."$$