First In-Class Exam Solutions
Math 246, Professor David Levermore
Thursday, 17 February 2011

(1) [5] Sketch the graph that you expect would be produced by the following MATLAB command.

```
>> ezplot('2/t', [1,6])
```

Solution. Your sketch should show a decreasing, concave up function that decreases from a value of 2 to a value of $\frac{1}{3}$ over the interval $[1,6]$.

(2) [20] Find the explicit solution for each of the following initial-value problems and identify its interval of definition.

(a) $\dfrac{\mathrm{d}u}{\mathrm{d}t} = -3t^2 e^{-u}\,, \quad u(2) = 0.$

Solution. This equation is separable. Its separated differential form is

$$e^u\,\mathrm{d}u = -3t^2\,\mathrm{d}t\,, \qquad \Longrightarrow \qquad e^u = -t^3 + c\,.$$

The initial condition $u(2) = 0$ implies that $c = e^0 + 2^3 = 1 + 8 = 9$. Therefore $e^u = -t^3 + 9$, which can be solved as

$$y = \log(9 - t^3)\,, \quad \text{with interval of definition } t < 9^{\frac{1}{3}}\,.$$

Here we need $9 > t^3$ for the log to be defined. The interval of definition is obtained by taking the cube root of both sides of this inequality.

(b) $\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\cos(t) - 2tv}{t^2}\,, \quad v(\pi) = 0.$

Solution. This equation is linear. Its linear normal form is

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{2}{t}\,v = \frac{\cos(t)}{t^2}\,.$$

An integrating factor is $\exp\bigl(\int_1^t \frac{2}{s}\,ds\bigr) = \exp(2\log(t)) = t^2$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl(t^2 v\bigr) = t^2 \cdot \frac{\cos(t)}{t^2} = \cos(t)\,, \qquad \Longrightarrow \qquad t^2 v = \sin(t) + c\,.$$

The initial condition $v(\pi) = 0$ implies that $c = \pi^2 \cdot 0 - \sin(\pi) = 0$. Therefore

$$v = \frac{\sin(t)}{t^2}\,, \quad \text{with interval of definition } 0 < t < \infty\,.$$

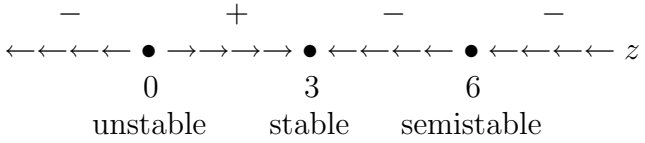
The interval of definition can be read off from the normal form of the equation because both the coefficient and forcing are continuous everywhere except at $t = 0$, where they are undefined, while the initial time $t = \pi$ is positive.

(3) [15] Consider the differential equation $\dfrac{\mathrm{d}z}{\mathrm{d}t} = z(3 - z)(6 - z)^2 e^z$.

(a) Sketch its phase-line portrait over the interval $-3 \le z \le 9$. Identify all of its stationary (equilibrium) solutions and classify each as being either stable, unstable, or semistable.

(b) If $z(0) = -2$, how does the solution $z(t)$ behave as $t \to \infty$?
(c) If $z(0) = 2$, how does the solution $z(t)$ behave as $t \to \infty$?
(d) If $z(0) = 8$, how does the solution $z(t)$ behave as $t \to \infty$?

Solution (a). The stationary solutions are $z = 0$, $z = 3$, and $z = 6$. A sign analysis of $z(3-z)(6-z)^2 e^z$ shows that the phase-line for this equation is therefore



Solution (b). The phase-line shows that if $z(0) = -2$ then $z(t) \to -\infty$ as $t \to \infty$.
Solution (c). The phase-line shows that if $z(0) = 2$ then $z(t) \to 3$ as $t \to \infty$.
Solution (d). The phase-line shows that if $z(0) = 8$ then $z(t) \to 6$ as $t \to \infty$.

(4) [10] Consider the following MATLAB function M-file.

```
function [t,y] = solveit(tI, yI, tF, n)

h = (tF - tI)/n;
hhalf = h/2;
t = zeros(n + 1, 1);
y = zeros(n + 1, 1);
t(1) = tI;
y(1) = yI;
for k = 1:n
thalf = t(k) + hhalf;
yhalf = y(k) + hhalf*((t(k))^2 - (y(k))^2);
t(k + 1) = t(k) + h;
y(k + 1) = y(k) + h*((thalf)^2 - (yhalf)^2);
end
```

Suppose the input values are $tI = 2$, $yI = 3$, $tF = 10$, and $n = 400$.
(a) What initial-value problem is being approximated numerically?
(b) What is the numerical method being used?
(c) What is the timestep?

Solution (a). The initial-value problem being approximated numerically is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t^2 - y^2\,, \qquad y(2) = 3\,.$$

Solution (b). The Heun-midpoint (modified Euler) method is being used. (This is clear from the lines defining thalf and yhalf.)

Solution (c). The time step is

$$h = \frac{\mathrm{tF} - \mathrm{tI}}{\mathrm{n}} = \frac{10-2}{400} = \frac{8}{400} = \frac{1}{50} = .02\,.$$

(5) [5] Suppose you have used the Runge-Kutta method to approximate the solution of an initial-value problem over the time interval $[0, 5]$ with 2000 uniform time steps. About how many uniform time steps would you need to reduce the global error of your approximation by a factor of $1/16$? (Give your reasoning.)

Solution. The Runge-Kutta is fourth order, so its global error scales like h^4. To reduce the error by a factor of $1/16$, you must reduce h by a factor of $(1/16)^{\frac{1}{4}} = 1/2$. You must therefore double the number of time steps, which means you need 4000 uniform time steps.

(6) [20] Give an implicit general solution to each of the following differential equations.

(a) $(2x\cos(y) + e^{2x})\,\mathrm{d}x + (y - x^2\sin(y))\,\mathrm{d}y = 0\,.$

Solution. This differential form is *exact* because

$$\partial_y(2x\cos(y) + e^{2x}) = -2x\sin(y) \quad = \quad \partial_x(y - x^2\sin(y)) = -2x\sin(y)\,.$$

We can therefore find $H(x, y)$ such that

$$\partial_x H(x, y) = 2x\cos(y) + e^{2x}\,, \qquad \partial_y H(x, y) = y - x^2\sin(y)\,.$$

Integrating the first equation with respect to x yields

$$H(x, y) = x^2\cos(y) + \tfrac{1}{2}e^{2x} + h(y)\,.$$

Plugging this expression for $H(x, y)$ into the second equation gives

$$-x^2\sin(y) + h'(y) = \partial_y H(x, y) = y - x^2\sin(y)\,,$$

which yields $h'(y) = y$. Take $h(y) = \frac{1}{2}y^2$, so that $H(x, y) = x^2\cos(y) + \frac{1}{2}e^{2x} + \frac{1}{2}y^2$. A general solution is therefore given implicitly by

$$x^2\cos(y) + \tfrac{1}{2}e^{2x} + \tfrac{1}{2}y^2 = c\,.$$

(b) $(3x^2y + 2xy + y^3)\,\mathrm{d}x + (x^2 + y^2)\,\mathrm{d}y = 0\,.$

Solution. This differential form is *not exact* because

$$\partial_y(3x^2y + 2xy + y^3) = 3x^2 + 2x + 3y^2 \quad \neq \quad \partial_x(x^2 + y^2) = 2x\,.$$

You therefore seek an *integrating factor* μ such that

$$\partial_y[(3x^2y + 2xy + y^3)\mu] = \partial_x[(x^2 + y^2)\mu]\,.$$

Expanding the partial derivatives by the product rule yields

$$(3x^2y + 2xy + y^3)\partial_y\mu + (3x^2 + 2x + 3y^2)\mu = (x^2 + y^2)\partial_x\mu + 2x\mu\,.$$

If set $\partial_y\mu = 0$ (thereby assuming that μ only depends upon x) then this becomes

$$(3x^2 + 2x + 3y^2)\mu = (x^2 + y^2)\partial_x\mu + 2x\mu\,,$$

which reduces to $\partial_y\mu = 3\mu$. This yields the integrating factor $\mu = e^{3x}$. Because e^{3x} is an integrating factor, the differential form

$$e^{3x}(3x^2y + 2xy + y^3)\,\mathrm{d}x + e^{3x}(x^2 + y^2)\,\mathrm{d}y = 0 \quad \text{is exact}\,.$$

You can therefore find $H(x, y)$ such that

$$\partial_x H(x, y) = e^{3x}(3x^2y + 2xy + y^3)\,, \qquad \partial_y H(x, y) = e^{3x}(x^2 + y^2)\,.$$

Integrating the second equation with respect to y yields

$$H(x,y) = e^{3x}(x^2y + \tfrac{1}{3}y^3) + h(x)\,.$$

Plugging this expression for $H(x,y)$ into the first equation gives

$$3e^{3x}(x^2y + \tfrac{1}{3}y^3) + e^{3x}2xy + h'(x) = \partial_x H(x,y) = e^{3x}(3x^2y + 2xy + y^3)\,,$$

which yields $h'(x) = 0$. Take $h(x) = 0$, so that $H(x,y) = e^{3x}(x^2y + \frac{1}{3}y^3)$. A general solution is therefore given implicitly by

$$e^{3x}(x^2y + \tfrac{1}{3}y^3) = c\,.$$

(7) [10] In the absence of predators the population of mosquitoes in a certain area would increase at a rate proportional to its current population and its population would triple every week. There are 90,000 mosquitoes in the area initially, and predators eat 80,000 mosquitoes per week. Write down an initial-value problem that governs the population of mosquitoes in the area at any time. (You do not have to solve the initial-value problem!)

Solution. Let $M(t)$ be the number of mosquitoes at time t weeks. Tripling every week means a growth like $3^t = e^{\log(3)t}$, which implies a growth rate of $\log(3)$ per mosquito. The rate at which the mosquitoes reproduce is thereby $\log(3)M(t)$ while the rate at which they are eaten is 80,000. The initial-value problem that M satisfies is therefore

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \log(3)M - 80,000\,, \qquad M(0) = 90,000\,.$$

(8) [5] Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{t^2-4}\,x = \frac{1}{\sin(t)}\,, \qquad x(1) = 0\,.$$

(You do not have to solve this equation to answer this question!)

Solution. This problem is linear in x and is already in normal form. The coefficient $1/(t^2-4)$ is continuous everywhere except where $t = \pm 2$, while the forcing $1/\sin(t)$ is continuous everywhere except where $t = n\pi$ for some integer n — i.e. everywhere except where $t = 0,\ \pm\pi,\ \pm 2\pi,\ \cdots$. You can therefore read off that the interval of definition is $(0,2)$, the endpoints of which are points where $1/\sin(t)$ and $1/(t^2-4)$ are undefined respectively that bracket the initial time $t = 1$.

(9) [10] Consider the initial-value problem

$$\frac{\mathrm{d}z}{\mathrm{d}t} = z^2 - z\,, \qquad z(0) = 2\,.$$

Use the Heun-trapezoidal (improved Euler) method with $h = .1$ to approximate $z(.1)$. (You can give your answer as an arithmetic expression.)

Solution. Set $z_0 = z(0) = 2$. The Heun-trapezoidal method then yields

$$\begin{aligned} f_0 &= z_0^2 - z_0 = 2^2 - 2 = 4 - 2 = 2\,, \\ \tilde{z}_1 &= z_0 + h\, f_0 = 2 + .1 \cdot 2 = 2.2\,, \\ \tilde{f}_1 &= \tilde{z}_1^2 - \tilde{z}_1 = (2.2)^2 - 2.2\,, \\ z(.1) \approx z_1 &= z_0 + \frac{h}{2}\,[f_0 + \tilde{f}_1] = 2 + .05\big(2 + (2.2)^2 - 2.2\big)\,. \end{aligned}$$

Because $(2.2)^2 = 4.84$, the optional arithmetic works out to

$$z(.1) \approx z_1 = 2 + .05 \cdot 4.64 = 2.232\,.$$