Second In-Class Exam Solutions
Math 246, Professor David Levermore
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(1) [6] Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}^4 y}{\mathrm{d}t^4} + \frac{7}{1-t^2}\,\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{5}{\sin(t)}\,, \qquad y(2) = y'(2) = y''(2) = y'''(2) = 0\,.$$

Solution. The equation is linear and in normal form. The coefficient is not defined at $t = \pm 1$ while the forcing is not defined at $t = n\pi$ for every integer n. The coefficient and forcing are both continuous over the interval $(1, \pi)$, which contains the initial time $t = 2$. The interval of definition is therefore $(1, \pi)$.

(2) [12] The functions e^t and te^t comprise a fundamental set of solutions to $y'' - 2y' + y = 0$.
(a) Find the solution $Y(t)$ to the general initial-value problem

$$y'' - 2y' + y = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,.$$

(b) Find the associated natural fundamental set of solutions to $y'' - 2y' + y = 0$.

Solution (a). Because e^t and te^t is a fundamental set of solutions to $y'' - 2y' + y = 0$, a general solution has the form $y(t) = c_1 e^t + c_2 te^t$. Then $y'(t) = c_1 e^t + c_2(e^t + te^t)$. To satisfy the initial conditions one needs

$$y_0 = y(0) = c_1\,, \qquad\qquad y_1 = y'(0) = c_1 + c_2\,.$$

You solve these equations to obtain $c_1 = y_0$ and $c_2 = y_1 - y_0$. The solution to the general initial-value problem is therefore

$$y(t) = y_0 e^t + (y_1 - y_0)te^t\,.$$

Solution (b). Because the above solution to the general initial-value problem can be written as

$$y(t) = (e^t - te^t)y_0 + te^t y_1\,,$$

the associated natural fundamental set of solutions is given by

$$N_0(t) = e^t - te^t\,, \qquad N_1(t) = te^t\,.$$

(3) [12] Give a general real solution of the equation

$$\mathrm{D}^2 y + 4\mathrm{D}y + 20y = 8\cos(2t)\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. This is a nonhomogeneous linear equation with constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 4z + 20 = (z+2)^2 + 16 = (z+2)^2 + 4^2\,,$$

which has the conjugate pair of roots $-2 \pm i4$. A general solution of the associated homogeneous equation is therefore

$$y_H(t) = c_1 e^{-2t}\cos(4t) + c_2 e^{-2t}\sin(4t)\,.$$

The forcing $8\cos(2t)$ has degree $d = 0$ and characteristic $\mu + i\nu = i2$, which is a root of $p(z)$ of multiplicity $m = 0$. A particular solution $y_P(t)$ can be found by the methods of either Key Identity Evaluations or Undetermined Coefficients.

Key Indentity Evaluations. Because $m = 0$ and $m + d = 0$, you only need to evaluate the Key Identity at $z = i2$, which yields

$$\mathrm{L}\big(e^{i2t}\big) = p(i2)e^{i2t} = \big((i2)^2 + 4(i2) + 20\big)e^{i2t} = (16 + i8)e^{i2t}\,.$$

Because the forcing has the form $8\cos(2t) = 8\,\mathrm{Re}(e^{i2t})$, we write

$$\mathrm{L}\left(\frac{e^{i2t}}{2+i}\right) = 8e^{i2t}\,,$$

which implies that

$$\begin{aligned} y_P(t) = \mathrm{Re}\left(\frac{e^{i2t}}{2+i}\right) &= \mathrm{Re}\left(\frac{e^{i2t}}{2+i}\,\frac{2-i}{2-i}\right) = \mathrm{Re}\left(\frac{(2-i)e^{i2t}}{2^2+1^2}\right) \\ &= \tfrac{1}{5}\,\mathrm{Re}\big((2-i)e^{i2t}\big) = \tfrac{1}{5}\big(2\cos(2t) + \sin(2t)\big)\,. \end{aligned}$$

A general solution of the equation is therefore

$$y(t) = c_1 e^{-2t}\cos(4t) + c_2 e^{-2t}\sin(4t) + \tfrac{2}{5}\cos(2t) + \tfrac{1}{5}\sin(2t)\,.$$

Undetermined Coefficients. Because $m = d = 0$, you *seek* a particular solution of the form

$$y_P(t) = A\cos(2t) + B\sin(2t)\,.$$

Because

$$y_P'(t) = -2A\sin(2t) + 2B\cos(2t)\,, \qquad y_P''(t) = -4A\cos(2t) - 4B\sin(2t)\,,$$

one sees that

$$\begin{aligned} \mathrm{L}y_P(t) &= y_P''(t) + 4y_P'(t) + 20y_P(t) \\ &= \big[-4A\cos(2t) - 4B\sin(2t)\big] + 4\big[-2A\sin(2t) + 2B\cos(2t)\big] \\ &\quad + 20\big[A\cos(2t) + B\sin(2t)\big] \\ &= (16A + 8B)\cos(2t) + (-8A + 16B)\sin(2t)\,. \end{aligned}$$

Setting $\mathrm{L}y_P(t) = 8\cos(2t)$, we see that

$$16A + 8B = 8\,, \qquad -8A + 16B = 0\,,$$

whereby $A = \frac{2}{5}$ and $B = \frac{1}{5}$. Hence, $y_P(t) = \frac{2}{5}\cos(2t) + \frac{1}{5}\sin(2t)$. A general solution of the equation is therefore

$$y(t) = c_1 e^{-2t}\cos(4t) + c_2 e^{-2t}\sin(4t) + \tfrac{2}{5}\cos(2t) + \tfrac{1}{5}\sin(2t)\,.$$

(4) [12] What answer will be produced by the following MATLAB commands?

```
>> ode = 'D2y + 7*Dy – 18*y = 33*exp(2*t)';
>> dsolve(ode, 't')

ans =
```

Solution. The commands ask MATLAB to give a general solution of the equation

$$\mathrm{D}^2y + 7\mathrm{D}y - 18y = 33e^{2t}\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

at least one version of MATLAB will produce the answer

```
exp(2*t)*C2 + exp(–9*t)*C1 + 3*exp(2*t)
```

Your answer does not have to be given in this MATLAB format. Rather, your answer should be equivalent to it. The fact this is the answer is seen as follows.

The problem being solved is a nonhomogeneous linear equation with constant coefficients. The characteristic polynomial is

$$p(z) = z^2 + 7z - 18 = (z+9)(z-2)\,,$$

which has the two real roots -9 and 2. A general solution of the associated homogeneous equation is

$$y_H(t) = c_1e^{-9t} + c_2e^{2t}\,.$$

The forcing $33e^{2t}$ has degree $d = 0$ and characteristic $\mu + i\nu = 2$, which is a root of $p(z)$ of multiplicity $m = 1$. A particular solution $y_P(t)$ can be found by the methods of either Key Identity Evaluation or Undetermined Coefficients.

Key Indentity Evaluations. Because $m = 1$ and $m + d = 1$, you need to evaluate the first derivative of the Key Identity at the characteristic $z = 2$. The Key Identity and its first derivative are

$$\begin{aligned}
\mathrm{L}\big(e^{zt}\big) &= \big(z^2 + 7z - 18\big)e^{zt}\,,\\
\mathrm{L}\big(t\,e^{zt}\big) &= \big(z^2 + 7z - 18\big)t\,e^{zt} + (2z+7)e^{zt}\,,
\end{aligned}$$

The first derivative of the Key Identity evaluated at $z = 2$ yields

$$\mathrm{L}\big(t\,e^{2t}\big) = (2\cdot 2 + 7)e^{2t} = 11e^{2t}\,.$$

Multiplying by 3 yields $\mathrm{L}(3te^{2t}) = 33e^{2t}$, which implies $y_P(t) = 3t\,e^{2t}$. A general solution is therefore

$$y(t) = c_1e^{-9t} + c_2e^{2t} + 3t\,e^{2t}\,.$$

Up to notational differences, this is the answer that MATLAB produces.

Undetermined Coefficients. Because $m = 1$ and $m + d = 1$, you *seek* a particular solution of the form

$$y_P(t) = At\,e^{2t}\,.$$

Because

$$y_P'(t) = 2At\,e^{2t} + Ae^{2t}\,, \qquad y_P''(t) = 4At\,e^{2t} + 4Ae^{2t}\,,$$

one sees that

$$\begin{aligned}
\mathrm{L}y_P(t) &= y_P''(t) + 7y_P'(t) - 18y_P(t)\\
&= \big[4At\,e^{2t} + 4Ae^{2t}\big] + 7\big[2At\,e^{2t} + Ae^{2t}\big] - 18\big[At\,e^{2t}\big]\\
&= 11Ae^{-2t}\,.
\end{aligned}$$

Setting $\mathrm{L}y_P(t) = 11Ae^{2t} = 33e^{2t}$, we see that $A = 3$. Hence, $y_P(t) = 3t\,e^{2t}$. A general solution is therefore

$$y(t) = c_1e^{-9t} + c_2e^{2t} + 3t\,e^{2t}\,.$$

Up to notational differences, this is the answer that MATLAB produces.

(5) [8] Compute the Green function associated with the differential operator

$$\mathrm{L} = \mathrm{D}^2 - 6\mathrm{D} + 10\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. The Green function $g(t)$ associated with the operator L satisfies the initial-value problem

$$\mathrm{L}g = \mathrm{D}^2g - 6\mathrm{D}g + 10g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

The characteristic polynomial is

$$p(z) = z^2 - 6z + 10 = (z-3)^2 + 1^2\,,$$

which has roots $3 \pm i$. Hence, the Green function has the form

$$g(t) = c_1e^{3t}\cos(t) + c_2e^{3t}\sin(t)\,.$$

The initial condition $g(0) = 0$ implies $c_1 = 0$. Because

$$g'(t) = 3c_2e^{3t}\sin(t) + c_2e^{3t}\cos(t)\,,$$

the initial condition $g'(0) = 1$ implies $c_2 = 1$. The Green function is thereby

$$g(t) = e^{3t}\sin(t)\,.$$

(6) [10] Find a particular solution $x_P(t)$ of the equation

$$\mathrm{D}^2x - 6\mathrm{D}x + 10x = \frac{e^{3t}}{\cos(t)}\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. This is a nonhomogeneous linear equation with constant coefficients. Its forcing does not have the characteristic form required for the methods of either Key Identity Evaluations or Undetermined Coefficients. You should therefore use either the Green Function method or Variation of Parameters.

Green Function. The equation is already in normal form. By the previous problem, the associated Green function is given by $g(t) = e^{3t}\sin(t)$. A particular solution is therefore

$$\begin{aligned}
x_P(t) &= \int_0^t e^{3(t-s)}\sin(t-s)\,\frac{e^{3s}}{\cos(s)}\,\mathrm{d}s \\
&= e^{3t}\int_0^t \big[\sin(t)\cos(s) - \cos(t)\sin(s)\big]\,\frac{1}{\cos(s)}\,\mathrm{d}s \\
&= e^{3t}\sin(t)\int_0^t \frac{\cos(s)}{\cos(s)}\,\mathrm{d}s - e^{3t}\cos(t)\int_0^t \frac{\sin(s)}{\cos(s)}\,\mathrm{d}s \\
&= e^{3t}\sin(t)\,t + e^{3t}\cos(t)\,\log(|\cos(t)|)\,.
\end{aligned}$$

Variation of Parameters. The equation is already in normal form. By the previous problem, a general solution of the associated homogeneous equation is

$$x_H(t) = c_1 e^{3t}\cos(t) + c_2 e^{3t}\sin(t)\,.$$

We therefore *seek* a solution of the nonhomogeneous equation in the form

$$x = u_1(t)e^{3t}\cos(t) + u_2(t)e^{3t}\sin(t)\,,$$

where $u_1'(t)$ and $u_2'(t)$ satisfy the linear algebraic system

$$u_1'(t)e^{3t}\cos(t) + u_2'(t)e^{3t}\sin(t) = 0\,,$$

$$u_1'(t)\big[3e^{3t}\cos(t) - e^{3t}\sin(t)\big] + u_2'(t)\big[3e^{3t}\sin(t) + e^{3t}\cos(t)\big] = \frac{e^{3t}}{\cos(t)}\,.$$

The solution of this system is

$$u_1'(t) = -\frac{\sin(t)}{\cos(t)}\,, \qquad u_2'(t) = 1\,.$$

Integrate these equations to obtain

$$u_1(t) = c_1 + \log(|\cos(t)|)\,, \qquad u_2(t) = c_2 + t\,.$$

A particular solution is therefore

$$x_P(t) = e^{3t}\cos(t)\,\log(|\cos(t)|) + e^{3t}\sin(t)\,t\,.$$

Remark. You can use the formulas for $u_1'(t)$ and $u_2'(t)$ given in the book, but this becomes more involved than simply setting up and solving the linear algebraic system as was done above.

Remark. The Green function method gets to the definite integrals quicker.

(7) [12] Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are $-1+i3$, $-1+i3$, $-1-i3$, $-1-i3$, $i2$, $-i2$, 4, 4, 4, -5, 0, 0.

(a) Give the order of L.

(b) Give a general real solution of the homogeneous equation $Ly = 0$.

Solution (a). There are 12 roots listed above, so the degree of the characteristic polynomial is 12, whereby the order of L is 12.

Solution (b). A general solution is

$$\begin{aligned} y(t) &= c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + c_3 t\,e^{-t}\cos(3t) + c_4 t\,e^{-t}\sin(3t) \\ &= c_5\cos(2t) + c_6\sin(2t) + c_7 e^{4t} + c_8 t e^{4t} + c_9 t^2 e^{4t} + c_{10}e^{-5t} + c_{11} + c_{12}t\,. \end{aligned}$$

The reasoning is as follows:

- the double conjugate pair $-1 \pm i3$ yields
$$e^{-t}\cos(3t)\,, \quad e^{-t}\sin(3t)\,, \quad t\,e^{-t}\cos(3t)\,, \quad \text{and} \quad t\,e^{-t}\sin(3t)\,;$$
- the single conjugate pair $\pm i2$ yields $\cos(2t)$ and $\sin(2t)$;
- the triple real root 4 yields e^{4t}, $t\,e^{4t}$, and $t^2 e^{4t}$;
- the single real root -5 yields e^{-5t};
- the double real root 0 yields 1 and t.

(8) [12] The functions t and t^2-1 are solutions of the homogeneous equation

$$(t^2+1)\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2t\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 0\,.$$

(You do not have to check that this is true!)

(a) Show they are linearly independent.

(b) Give a general solution of the equation

$$(t^2+1)\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2t\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = (t^2+1)^2\,.$$

Solution (a). The Wronskian of t and t^2-1 is

$$W[t,t^2-1](t) = \det\begin{pmatrix} t & t^2-1 \\ 1 & 2t \end{pmatrix} = 2t^2 - (t^2-1) = t^2+1\,.$$

Because $W[t,t^2-1](t) = t^2+1 \neq 0$, the functions t and t^2-1 are linearly independent.

Alternative Solution (a). Suppose that

$$c_1 t + c_2(t^2-1) = 0\,.$$

By setting $t=1$ and $t=0$ into this relation gives $c_1=0$ and $c_2=0$ respectively. Therefore the functions t and t^2-1 are linearly independent.

Solution (b). Because t and t^2-1 are linearly independent, a general solution of the associated homogeneous problem is

$$y_H(t) = c_1 t + c_2(t^2-1)\,.$$

Because this problem has variable coefficients, you should use either the general Green Function method or Variation of Parameters to find a particular solution $y_P(t)$. In either case you should first divide by t^2+1 to bring the equation into its normal form

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{2t}{t^2+1}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{2}{t^2+1}y = t^2+1\,.$$

General Green Function. The Green function $G(t,s)$ is given by

$$G(t,s) = \frac{\det\begin{pmatrix} s & s^2-1 \\ t & t^2-1 \end{pmatrix}}{W[s,s^2-1](s)} = \frac{s(t^2-1)-t(s^2-1)}{s^2+1} = \frac{(t^2-1)s - t(s^2-1)}{s^2+1}\,.$$

The Green function formula then yields the solution

$$\begin{aligned} y_P(t) &= \int_0^t G(t,s)\,(s^2+1)\,\mathrm{d}s \\ &= \int_0^t \frac{(t^2-1)s - t(s^2-1)}{s^2+1}\,(s^2+1)\,\mathrm{d}s \\ &= (t^2-1)\int_0^t s\,\mathrm{d}s - t\int_0^t (s^2-1)\,\mathrm{d}s \\ &= (t^2-1)\tfrac{1}{2}t^2 - t\big(\tfrac{1}{3}t^3 - t\big) = \tfrac{1}{6}t^4 + \tfrac{1}{2}t^2\,. \end{aligned}$$

A general solution is therefore

$$y(t) = c_1 t + c_2(t^2-1) + \tfrac{1}{6}t^4 + \tfrac{1}{2}t^2\,.$$

Variation of Parameters. *Seek* a solution in the form

$$y = u_1(t)t + u_2(t)(t^2 - 1)\,.$$

where $u_1'(t)$ and $u_2'(t)$ satisfy the linear algebraic system

$$u_1'(t)t + u_2'(t)(t^2 - 1) = 0\,,$$
$$u_1'(t)1 + u_2'(t)2t = t^2 + 1\,.$$

The solution of this system is

$$u_1'(t) = -(t^2 - 1)\,, \qquad u_2'(t) = t\,.$$

Alternatively, because you know that $W[t, t^2 - 1](t) = t^2 + 1$, you can directly use the formulas from the book to obtain

$$u_1'(t) = \frac{-(t^2 - 1)\cdot(t^2 + 1)}{t^2 + 1} = -(t^2 - 1)\,, \qquad u_2'(t) = \frac{t\cdot(t^2 + 1)}{t^2 + 1} = t\,.$$

No matter how they are obtained, you integrate these equations to find

$$u_1(t) = c_1 - \tfrac{1}{3}t^3 + t\,, \qquad u_2(t) = c_2 + \tfrac{1}{2}t^2\,.$$

A general solution is therefore

$$\begin{aligned} y &= \big(c_1 - \tfrac{1}{3}t^3 + t\big)t + \big(c_2 + \tfrac{1}{2}t^2\big)(t^2 - 1) \\ &= c_1 t - \tfrac{1}{3}t^4 + t^2 + c_2(t^2 - 1) + \tfrac{1}{2}t^4 - \tfrac{1}{2}t^2 \\ &= c_1 t + c_2(t^2 - 1) + \tfrac{1}{6}t^4 + \tfrac{1}{2}t^2\,. \end{aligned}$$

(9) [8] When a .5 kilogram mass is hung vertically from a spring, at rest it stretches the spring .05 m. (Gravitational acceleration is $g = 9.8$ m/sec^2.) At $t = 0$ the mass is displaced .2 m below its rest position and is released with a downward initial velocity of .3 m/sec. Assume that the spring force is proportional to displacement, that there is no damping, and that the mass is driven by an external force of $F_{ext}(t) = 5\cos(\omega t)$ Newtons (1 Newton = 1 kg m/sec^2), where up is taken to be positive.

(a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)
(b) What is the natural frequency ω_o of this spring?
(c) At what value of the driving frequency ω does resonance occur?

Solution (a). Let $h(t)$ be the displacement (in meters) of the mass from its rest position at time t (in seconds), with upward displacements being positive. The governing initial-value problem then has the form

$$m\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + kh = F_{ext}(t)\,, \qquad h(0) = -.2\,, \quad h'(0) = -.3\,,$$

where m is the mass and k is the spring constant. The problem says that $m = .5$ kilograms. The spring constant is obtained by balancing the weight of the mass ($mg = .5 \cdot 9.8$ Newtons) with the force applied by the spring when it is stetched .05 m. This gives $k\,.05 = .5 \cdot 9.8$, or

$$k = \frac{.5 \cdot 9.8}{.05} = 10 \cdot 9.8 = 98 \quad \text{Newtons/m}\,.$$

Because $F_{ext}(t) = 5\cos(\omega t)$, the governing initial-value problem is therefore

$$.5\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + 98\,h = 5\cos(\omega t)\,, \qquad h(0) = -.2\,, \quad h'(0) = -.3\,.$$

While the above answer was sufficient, had you put the initial-value problem into normal form you should have obtained

$$\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + 2\cdot 98\,h = 10\cos(\omega t)\,, \qquad h(0) = -.2\,, \quad h'(0) = -.3\,.$$

Solution (b). The natural frequency of the spring is given by

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{9.8}{.05}} = \sqrt{2\cdot 98} = 14 \quad 1/\mathrm{sec}\,.$$

Solution (c). Resonance occurs when the driving frequency ω equals the natural fequency of the spring ω_o. Given the answer to part (b), resonance occurs when

$$\omega = \omega_o = 14 \quad 1/\mathrm{sec}\,.$$

(10) [8] The vertical displacement of a mass on a spring is given by

$$h(t) = -12e^{-t}\cos(2t) + 5e^{-t}\sin(2t)\,.$$

(a) Why is this system under damped?
(b) Express $h(t)$ in the form $h(t) = Ae^{-t}\cos(\nu t - \delta)$ with $A > 0$ and $0 \le \delta < 2\pi$, identifying the quasiperiod and phase of the oscillation. (The phase may be expressed in terms of an inverse trig function.)

Solution (a). The system is under damped because the given displacement $h(t)$ comes from an underlying characteristic polynomial having the complex conjugate pair of roots $-1 \pm i2$.

Solution (b). By compairing

$$Ae^{-t}\cos(\nu t - \delta) = Ae^{-t}\cos(\delta)\cos(\nu t) + Ae^{-t}\sin(\delta)\sin(\nu t)\,,$$

with $h(t) = -12e^{-t}\cos(2t) + 5e^{-t}\sin(2t)$, we see that $\nu = 2$ and that

$$A\cos(\delta) = -12\,, \qquad A\sin(\delta) = 5\,.$$

This shows that (A, δ) are the polar coordinates of the point in the plane whose Cartesian coordinates are $(-12, 5)$. Clearly A is given by

$$A = \sqrt{(-12)^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13\,.$$

Because $(-12, 5)$ lies in the second quadrant, the phase δ satisfies $\frac{\pi}{2} < \delta < \pi$. Because

$$\sin(\delta) = \tfrac{5}{13}\,, \qquad \tan(\delta) = -\tfrac{5}{12}\,, \qquad \cos(\delta) = -\tfrac{12}{13}\,,$$

you can express the phase by any one of the formulas

$$\delta = \pi - \sin^{-1}\!\left(\tfrac{5}{13}\right), \qquad \delta = \pi - \tan^{-1}\!\left(\tfrac{5}{12}\right), \qquad \delta = \pi - \cos^{-1}\!\left(\tfrac{12}{13}\right).$$

Finally, the quasiperiod T is given by

$$T = \frac{2\pi}{\nu} = \frac{2\pi}{2} = \pi\,.$$