Third In-Class Exam Solutions
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(1) [6] Given that 3 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 4 & 1 \\ 1 & 0 & 2 \end{pmatrix} ,$$

find all the eigenvectors of $\mathbf{A}$ associated with 3.

Solution. The eigenvectors of $\mathbf{A}$ associated with 3 are all nonzero vectors $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = 3\mathbf{v}$. Equivalently, they are all nonzero vectors $\mathbf{v}$ such that $(\mathbf{A} - 3\mathbf{I})\mathbf{v} = \mathbf{0}$, which is

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

The entries of $\mathbf{v}$ thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned} v_2 + 2v_3 &= 0 \,, \\ v_1 + v_2 + v_3 &= 0 \,, \\ v_1 \qquad - v_3 &= 0 \,. \end{aligned}$$

You may solve this system either by elimination or by row reduction. By either method you find that its general solution is

$$v_1 = \alpha \,, \quad v_2 = -2\alpha \,, \quad v_3 = \alpha \,, \qquad \text{for any constant } \alpha \,.$$

The eigenvectors of $\mathbf{A}$ associated with 3 therefore have the form

$$\alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{for any constant } \alpha \neq 0 \,.$$

(2) [8] A 3×3 matrix $\mathbf{A}$ has the eigenpairs

$$\left(-1 \,, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right) , \qquad \left(2 \,, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right) , \qquad \left(5 \,, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right) .$$

(a) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $e^{t\mathbf{A}} = \mathbf{V} e^{t\mathbf{D}} \mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $e^{t\mathbf{A}}$!)

Solution. One choice for $\mathbf{V}$ and $\mathbf{D}$ is

$$\mathbf{V} = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} , \qquad \mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} .$$

(b) Give a fundamental matrix for the system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$.

Solution. Use the given eigenpairs to construct the solutions

$$\mathbf{x}_1(t) = e^{-t}\begin{pmatrix}1\\2\\1\end{pmatrix}, \quad \mathbf{x}_2(t) = e^{2t}\begin{pmatrix}-2\\1\\1\end{pmatrix}, \quad \mathbf{x}_3(t) = e^{5t}\begin{pmatrix}1\\0\\-1\end{pmatrix},$$

Then a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = \begin{pmatrix}\mathbf{x}_1(t) & \mathbf{x}_2(t) & \mathbf{x}_3(t)\end{pmatrix} = \begin{pmatrix}e^{-t} & -2e^{2t} & e^{5t}\\ 2e^{-t} & e^{2t} & 0\\ e^{-t} & e^{2t} & -e^{5t}\end{pmatrix}.$$

Alternative Solution. Given the $\mathbf{V}$ and $\mathbf{D}$ from part (a), a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = \mathbf{V}e^{t\mathbf{D}} = \begin{pmatrix}1 & -2 & 1\\ 2 & 1 & 0\\ 1 & 1 & -1\end{pmatrix}\begin{pmatrix}e^{-t} & 0 & 0\\ 0 & e^{2t} & 0\\ 0 & 0 & e^{5t}\end{pmatrix} = \begin{pmatrix}e^{-t} & -2e^{2t} & e^{5t}\\ 2e^{-t} & e^{2t} & 0\\ e^{-t} & e^{2t} & -e^{5t}\end{pmatrix}.$$

(3) [4] Given that $e^{t\mathbf{A}} = \begin{pmatrix}\cos(3t) & -\frac{1}{3}\sin(3t)\\ 3\sin(3t) & \cos(3t)\end{pmatrix}$, solve the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\y\end{pmatrix} = \mathbf{A}\begin{pmatrix}x\\y\end{pmatrix}, \qquad \begin{pmatrix}x(0)\\y(0)\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix}.$$

Solution. The solution of the initial-value problem is given by

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix}1\\-3\end{pmatrix} = \begin{pmatrix}\cos(3t) & -\frac{1}{3}\sin(3t)\\ 3\sin(3t) & \cos(3t)\end{pmatrix}\begin{pmatrix}1\\-3\end{pmatrix} = \begin{pmatrix}\cos(3t)+\sin(3t)\\ 3\sin(3t)-3\cos(3t)\end{pmatrix}.$$

(4) [16] Find a general solution for each of the following systems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3 & 3\\ 4 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix}3 & 3\\ 4 & -1\end{pmatrix}$ is given by

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 15 = (z+3)(z-5).$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -3 and 5. Because

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix}6 & 3\\ 4 & 2\end{pmatrix}, \qquad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix}-2 & 3\\ 4 & -6\end{pmatrix},$$

you can read off that eigenpairs are

$$\left(-3, \begin{pmatrix}-1\\2\end{pmatrix}\right), \qquad \left(5, \begin{pmatrix}3\\2\end{pmatrix}\right).$$

A general solution of the system is therefore

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 3 \\ 2 \end{pmatrix} .$$

Alternative Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 15 = (z-1)^2 - 4^2 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 1 ± 4. Because

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\cosh(4t)\mathbf{I} + \frac{\sinh(4t)}{4}(\mathbf{A} - \mathbf{I}) \right] \\ &= e^t \left[\cosh(4t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(4t)}{4} \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} \cosh(4t) + \frac{1}{2}\sinh(4t) & \frac{3}{4}\sinh(4t) \\ \sinh(4t) & \cosh(4t) - \frac{1}{2}\sinh(4t) \end{pmatrix} , \end{aligned}$$

a general solution is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{c} = c_1 e^t \begin{pmatrix} \cosh(4t) + \frac{1}{2}\sinh(4t) \\ \sinh(4t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \frac{3}{4}\sinh(4t) \\ \cosh(4t) - \frac{1}{2}\sinh(4t) \end{pmatrix} .$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 7 & -1 \\ 4 & 3 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 10z + 25 = (z-5)^2 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which is only 5. Because

$$\begin{aligned} e^{t\mathbf{A}} &= e^{5t} \left[\mathbf{I} + t\,(\mathbf{A} - 5\mathbf{I}) \right] \\ &= e^{5t} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \right] = e^{5t} \begin{pmatrix} 1+2t & -t \\ 4t & 1-2t \end{pmatrix} , \end{aligned}$$

a general solution is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{c} = c_1 e^{5t} \begin{pmatrix} 1+2t \\ 4t \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} -t \\ 1-2t \end{pmatrix} .$$

Alternative Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 7 & -1 \\ 4 & 3 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 10z + 25 = (z-5)^2 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which is only 5. Because

$$\mathbf{A} - 5\mathbf{I} = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} ,$$

you can read off that one eigenpair is

$$\left(5\,,\,\begin{pmatrix}1\\2\end{pmatrix}\right).$$

You can use this eigenpair to construct the solution

$$\mathbf{x}_1(t) = e^{5t}\begin{pmatrix}1\\2\end{pmatrix}.$$

A second solution can be constructed by the formula

$$\mathbf{x}_2(t) = e^{5t}\mathbf{w} + t\,e^{5t}(\mathbf{A} - 5\mathbf{I})\mathbf{w}\,,$$

where $\mathbf{w}$ is any nonzero vector that is *not* an eigenvector for the eigenvalue 5. For example, you can set

$$\mathbf{x}_2(t) = e^{5t}\begin{pmatrix}1\\0\end{pmatrix} + t\,e^{5t}\begin{pmatrix}2 & -1\\4 & -2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = e^{5t}\begin{pmatrix}1\\0\end{pmatrix} + t\,e^{5t}\begin{pmatrix}2\\4\end{pmatrix} = e^{5t}\begin{pmatrix}1+2t\\4t\end{pmatrix}.$$

A general solution is then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1e^{5t}\begin{pmatrix}1\\2\end{pmatrix} + c_2e^{5t}\begin{pmatrix}1+2t\\4t\end{pmatrix}.$$

(5) [12] Sketch phase-plane portraits for *both* of the systems in the previous problem. For each portrait identify its type and give a reason why the origin is either attracting, stable, unstable, or repelling.

Solution (a). Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z+3)(z-5)$, its eigenvalues are -3 and 5. Because these are real with opposite signs, the phase portrait is a *saddle*. The origin is thereby *unstable*, but not repelling. Because

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix}6 & 3\\4 & 2\end{pmatrix}, \qquad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix}-2 & 3\\4 & -6\end{pmatrix},$$

you can read off that eigenpairs are

$$\left(-3\,,\,\begin{pmatrix}-1\\2\end{pmatrix}\right), \qquad \left(5\,,\,\begin{pmatrix}3\\2\end{pmatrix}\right).$$

The solutions associated with the eigenvalue -3 will lie on the line $y = -2x$ and move toward the origin. The solutions associated with the eigenvalue 5 will lie on the line $y = \frac{2}{3}x$ and move away from the origin. Every other trajectory will sweep in moving away from the line $y = -2x$ and sweep out moving toward the line $y = \frac{2}{3}x$.

Solution (b). Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z-5)^2$, one sees that $\mu = 5$ and $\delta = 0$. Because $\mu = 5 > 0$, $\delta = 0$, and $a_{21} = 4 > 0$ the phase portrait is a *counterclockwise twist source*. The origin is thereby *repelling* (and also *unstable*). Because

$$\mathbf{A} - 5\mathbf{I} = \begin{pmatrix}2 & -1\\4 & -2\end{pmatrix},$$

we see that the eigenvectors associated with 5 have the form

$$\alpha\begin{pmatrix}1\\2\end{pmatrix} \quad \text{for some } \alpha \neq 0\,.$$

The associated solutions will therefore lie on the line $y = 2x$ and move away from the origin. Every other trajectory emerges from the origin tangent to the line $y = 2x$ and moves away with a counterclockwise twist.

(6) [8] Consider two interconnected tanks filled with brine (salt water). The first tank contains 72 liters and the second contains 55 liters. Brine with a concentration of 13 grams of salt per liter flows into the first tank at a rate of 8 liters per hour. Well stirred brine flows from the first tank into the second at a rate of 7 liters per hour, from the second into the first at a rate of 4 liters per hour, from the first into a drain at a rate of 5 liter per hour, and from the second into a drain at a rate of 3 liters per hour. At $t = 0$ there are 125 grams of salt in the first tank and 40 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution. The rates work out so there will always be 72 liters of brine in the first tank and 55 liters in the second. Let $S_1(t)$ be the grams of salt in the first tank and $S_2(t)$ be the grams of salt in the second tank. These are governed by the initial-value problem

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 13{\cdot}8 + \frac{S_2}{55}\,4 - \frac{S_1}{72}\,7 - \frac{S_1}{72}\,5\,, \qquad & S_1(0) &= 125\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{72}\,7 - \frac{S_2}{55}\,4 - \frac{S_2}{55}\,3\,, & S_2(0) &= 40\,. \end{aligned}$$

You could leave the answer in the above form. It can however be simplified to

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 104 + \frac{S_2}{55}\,4 - \frac{S_1}{6}\,, \qquad & S_1(0) &= 125\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{72}\,7 - \frac{S_2}{55}\,7\,, & S_2(0) &= 40\,. \end{aligned}$$

(7) [6] Transform the equation $v'''' - t^2 v''' + v'v'' - e^t v = \sin(5t)$ into a first-order system of ordinary differential equations.

Solution. Because the equation is fourth order, the first order system must have dimension four. The simplest such first order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ \sin(5t) + e^t x_1 - x_2 x_3 + t^2 x_4 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} v \\ v' \\ v'' \\ v''' \end{pmatrix}.$$

(8) [13] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^3 \\ 3 + t^5 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

Solution. The Wronskian is

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} 1 & t^3 \\ t^2 & 3 + t^5 \end{pmatrix} = 3 + t^5 - t^5 = 3\,.$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

Solution. Let $\mathbf{\Psi}(t) = \begin{pmatrix} 1 & t^3 \\ t^2 & 3+t^5 \end{pmatrix}$. Because $\dfrac{\mathrm{d}\mathbf{\Psi}(t)}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{\Psi}(t)$, one has

$$\begin{aligned}\mathbf{A}(t) = \frac{\mathrm{d}\mathbf{\Psi}(t)}{\mathrm{d}t}\mathbf{\Psi}(t)^{-1} &= \begin{pmatrix} 0 & 3t^2 \\ 2t & 5t^4 \end{pmatrix}\begin{pmatrix} 1 & t^3 \\ t^2 & 3+t^5 \end{pmatrix}^{-1} \\ &= \frac{1}{3}\begin{pmatrix} 0 & 3t^2 \\ 2t & 5t^4 \end{pmatrix}\begin{pmatrix} 3+t^5 & -t^3 \\ -t^2 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -3t^4 & 3t^2 \\ 6t-3t^6 & 3t^4 \end{pmatrix} = \begin{pmatrix} -t^4 & t^2 \\ 2t-t^6 & t^4 \end{pmatrix}.\end{aligned}$$

(c) Find the natural fundamental matrix associated with the initial time 0 for the system you found in part (b).

Solution. Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions to the system found in part (b), a fundamental is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} 1 & t^3 \\ t^2 & 3+t^5 \end{pmatrix}.$$

The natural fundamental matrix associated with the initial time 0 is therefore

$$\begin{aligned}\mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} &= \begin{pmatrix} 1 & t^3 \\ t^2 & 3+t^5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & t^3 \\ t^2 & 3+t^5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{3}t^3 \\ t^2 & 1+\frac{1}{3}t^5 \end{pmatrix}.\end{aligned}$$

(9) [8] Compute the Laplace transform of $f(t) = u(t-3)\,e^{2t}$ from its definition. (Here u is the unit step function.)

Solution. The definition of Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\int_0^T e^{-st}u(t-3)\,e^{2t}\,\mathrm{d}t = \lim_{T\to\infty}\int_3^T e^{-(s-2)t}\,\mathrm{d}t\,.$$

When $s \leq 2$ this limit diverges to $+\infty$ because in that case one has for every $T > 3$ that

$$\int_3^T e^{-(s-2)t}\,\mathrm{d}t \geq \int_3^T \mathrm{d}t = T-3\,,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

When $s > 2$ one has for every $T > 3$ that

$$\int_3^T e^{-(s-2)t}\,\mathrm{d}t = -\frac{e^{-(s-2)t}}{s-2}\bigg|_3^T = -\frac{e^{-(s-2)T}}{s-2} + \frac{e^{-(s-2)3}}{s-2}\,,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\left[-\frac{e^{-(s-2)T}}{s-2} + \frac{e^{-(s-2)3}}{s-2}\right] = \frac{e^{-(s-2)3}}{s-2}\,.$$

(10) [10] Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' - 6y' + 13y = f(t)\,, \qquad y(0) = 5\,, \quad y'(0) = -2\,,$$

where

$$f(t) = \begin{cases} \frac{1}{4}t^2 & \text{for } 0 \le t < 2\,, \\ e^{2-t} & \text{for } t \ge 2\,. \end{cases}$$

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) - 6\mathcal{L}[y'](s) + 13\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s) - 5\,, \\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 5s + 2\,. \end{aligned}$$

To compute $\mathcal{L}[f](s)$, first write f as

$$\begin{aligned} f(t) &= \big(u(t) - u(t-2)\big)\tfrac{1}{4}t^2 + u(t-2)e^{2-t} \\ &= u(t)\tfrac{1}{4}t^2 + u(t-2)\big(e^{2-t} - \tfrac{1}{4}t^2\big) \\ &= \tfrac{1}{4}t^2 + u(t-2)j(t-2)\,, \end{aligned}$$

where

$$j(t) = e^{2-(t+2)} - \tfrac{1}{4}(t+2)^2 = e^{-t} - \tfrac{1}{4}t^2 - t - 1\,.$$

Referring to the table on the last page, item 6 with $c = 2$ and $j(t) = e^{-t} - \frac{1}{4}t^2 - t - 1$ shows that

$$\mathcal{L}\big[u(t-2)j(t-2)\big](s) = e^{-2s}\mathcal{L}[j](s) = e^{-2s}\mathcal{L}\big[e^{-t} - \tfrac{1}{4}t^2 - t - 1\big](s)\,,$$

while item 1 with $a = -1$ and $n = 0$, with $a = 0$ and $n = 2$, with $a = 0$ and $n = 1$, and with $a = 0$ and $n = 0$ shows that

$$\mathcal{L}[e^{-t}](s) = \frac{1}{s+1}\,, \qquad \mathcal{L}[t^2](s) = \frac{2}{s^3}\,, \qquad \mathcal{L}[t](s) = \frac{1}{s^2}\,, \qquad \mathcal{L}[1](s) = \frac{1}{s}\,,$$

whereby

$$\mathcal{L}\big[e^{-t} - \tfrac{1}{4}t^2 - t - 1\big](s) = \frac{1}{s+1} - \frac{1}{2s^3} - \frac{1}{s^2} - \frac{1}{s}\,.$$

Therefore

$$\begin{aligned} \mathcal{L}[f](s) &= \mathcal{L}\big[\tfrac{1}{4}t^2 + u(t-2)j(t-2)\big](s) \\ &= \frac{1}{2s^3} + e^{-2s}\left(\frac{1}{s+1} - \frac{1}{2s^3} - \frac{1}{s^2} - \frac{1}{s}\right). \end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2Y(s) - 5s + 2\big) - 6\big(sY(s) - 5\big) + 13Y(s) = \frac{1 - e^{-2s}}{2s^3} + \frac{e^{-2s}}{s+1} - \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s}\,,$$

which becomes

$$(s^2 - 6s + 13)Y(s) - 5s - 32 = \frac{1 - e^{-2s}}{2s^3} + \frac{e^{-2s}}{s+1} - \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s}\,.$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 - 6s + 13}\left(5s + 32 + \frac{1 - e^{-2s}}{2s^3} + \frac{e^{-2s}}{s+1} - \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s}\right).$$

(11) [9] Find the inverse Laplace transforms of the function

$$F(s) = e^{-5s}\,\frac{s+10}{s^2 + 2s - 8}\,.$$

You may refer to the table on the last page.

Solution. The denominator factors as $(s-2)(s+4)$, so the partial fraction decomposition is

$$\frac{s+10}{s^2+2s-8} = \frac{s+10}{(s-2)(s+4)} = \frac{2}{s-2} + \frac{-1}{s+4}\,.$$

Referring to the table on the last page, item 1 with $a = 2$ and $n = 0$, and with $a = -4$ and $n = 0$ shows that

$$\mathcal{L}[e^{2t}](s) = \frac{1}{s-2}\,, \qquad \mathcal{L}[e^{-4t}](s) = \frac{1}{s+4}\,.$$

You therefore obtain

$$\mathcal{L}^{-1}\left[\frac{s+10}{s^2+2s-8}\right](t) = 2\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+4}\right] = 2e^{2t} - e^{-4t}\,.$$

Then item 6 with $c = 5$ and $j(t) = 2e^{2t} - e^{-4t}$ shows that

$$\begin{aligned}\mathcal{L}^{-1}\left[e^{-5s}\frac{s+10}{s^2+2s-8}\right](t) &= u(t-5)\,\mathcal{L}^{-1}\left[\frac{s+10}{s^2+2s-8}\right](t-4)\\ &= u(t-5)\left(2e^{2(t-5)} - e^{-4(t-5)}\right).\end{aligned}$$

A Short Table of Laplace Transforms

$\mathcal{L}[t^n e^{at}](s) = \dfrac{n!}{(s-a)^{n+1}}$	for $s > a$.
$\mathcal{L}[e^{at}\cos(bt)](s) = \dfrac{s-a}{(s-a)^2 + b^2}$	for $s > a$.
$\mathcal{L}[e^{at}\sin(bt)](s) = \dfrac{b}{(s-a)^2 + b^2}$	for $s > a$.
$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s)$	where $J(s) = \mathcal{L}[j(t)](s)$.
$\mathcal{L}[e^{at} j(t)](s) = J(s-a)$	where $J(s) = \mathcal{L}[j(t)](s)$.
$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s)$	where $J(s) = \mathcal{L}[j(t)](s)$ and u is the unit step function.