Final Exam Sample Problems, Math 246, Spring 2011

(1) Consider the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}t} = (9 - y^2)y^2$.
   (a) Identify its equilibrium (stationary) points and classify their stability.
   (b) Sketch how solutions move in the interval $-5 \le y \le 5$ (its phase-line portrait).
   (c) If $y(0) = -1$, how does the solution $y(t)$ behave as $t \to \infty$?

(2) Solve (possibly implicitly) each of the following initial-value problems. Identify their intervals of definition.
   (a) $\dfrac{\mathrm{d}y}{\mathrm{d}t} + \dfrac{2ty}{1+t^2} = t^2\,, \quad y(0) = 1\,.$
   (b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{e^x y + 2x}{2y + e^x} = 0\,, \quad y(0) = 0\,.$

(3) Consider the following MATLAB function M-file.

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function [t,y] = solveit(ti, yi, tf, n)
t = zeros(n + 1, 1); y = zeros(n + 1, 1);
t(1) = ti; y(1) = yi; h = (tf - ti)/n;
for i = 1:n
t(i + 1) = t(i) + h; y(i + 1) = y(i) + h*((t(i))^4 + (y(i))^2);
end
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   Suppose that the input values are ti = 1, yi = 1, tf = 5, and n = 40.
   (a) What is the initial-value problem being approximated numerically?
   (b) What is the numerical method being used?
   (c) What is the timestep (step size)?
   (d) What are the output values for the first two approximations, y(2) and y(3)?

(4) Give an explicit real-valued general solution of the following equations.
   (a) $y'' - 2y' + 5y = te^t + \cos(2t)$
   (b) $u'' - 3u' - 10u = te^{-2t}$
   (c) $v'' + 9v = \cos(3t)$

(5) Solve the following initial-value problems.
   (a) $w'' + 4w' + 20w = 5e^{2t}, \qquad w(0) = 3\,, \quad w'(0) = -7.$
   (b) $y'' - 4y' + 4y = \dfrac{e^{2t}}{3+t}, \qquad y(0) = 0\,, \quad y'(0) = 5.$
   You should try to evaluate any definite integrals that arise.

(6) When a mass of 2 kilograms is hung vertically from a spring, it stretches the spring 0.5 meters. (Gravitational acceleration is 9.8 m/sec^2.) At $t = 0$ the mass is set in motion from 0.3 meters below its equilibrium (rest) position with a upward velocity of 2 m/sec. It is acted upon by an external force of $2\cos(5t)$. Neglect drag and assume that the spring force is proportional to its displacement. Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem; just write it down!)

(7) Give an explicit general solution of the equation

$$h'' + 2h' + 5h = 0\,.$$

Sketch a typical solution for $t \geq 0$. If this equation governs a damped spring-mass system, is the system over, under, or critically damped?

(8) Find the Laplace transform $Y(s)$ of the solution $y(t)$ to the initial-value problem

$$y'' + 4y' + 8y = f(t)\,, \qquad y(0) = 2\,, \quad y'(0) = 4\,.$$

where

$$f(t) = \begin{cases} 4 & \text{for } 0 \leq t < 2\,, \\ t^2 & \text{for } 2 \leq t\,. \end{cases}$$

You may refer to the Laplace table in Section 6.2 of the book. (DO NOT take the inverse Laplace transform to find $y(t)$; just solve for $Y(s)$!)

(9) Find the function $y(t)$ whose Laplace transform $Y(s)$ is given by

$$\text{(a)} \quad Y(s) = \frac{e^{-3s}4}{s^2 - 6s + 5}\,, \qquad\qquad \text{(b)} \quad Y(s) = \frac{e^{-2s}s}{s^2 + 4s + 8}\,.$$

You may refer to the Laplace table in Section 6.2 of the book.

(10) Consider two interconnected tanks filled with brine (salt water). The first tank contains 80 liters and the second contains 30 liters. Brine flows with a concentration of 3 grams of salt per liter flows into the first tank at a rate of 2 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 6 liters per hour, from the second to the first at a rate of 4 liters per hour, and from the second into a drain at a rate of 2 liters per hour. At $t = 0$ there are 7 grams of salt in the first tank and 25 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(11) Consider the real vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^3 \\ 3 + t^4 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.
(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the linear system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$.
(c) Give a general solution to the system you found in part (b).

(12) Give a general real vector-valued solution of the linear planar system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for

$$\text{(a)} \quad \mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}\,, \qquad\qquad \text{(b)} \quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\,.$$

(13) What answer will be produced by the following MATLAB command?

$$>> \text{A} = [1\ 4;\ 3\ 2];\ [\text{vect, val}] = \text{eig(sym(A))}$$

You do not have to give the answer in MATLAB format.

(14) A real 2×2 matrix $\mathbf{A}$ has eigenvalues 2 and -1 with associated eigenvectors

$$\begin{pmatrix}3\\1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}-1\\2\end{pmatrix}.$$

(a) Give a general solution to the linear planar system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
(b) Compute $e^{t\mathbf{A}}$.
(c) Sketch a phase-plane portrait for this system and identify its type. Classify the stability of the origin. (Carefully mark all sketched trajectories with arrows!)

(15) Solve the initial-value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_I$ and describe how its solution behaves as $t \to \infty$ for the following $\mathbf{A}$ and $\mathbf{x}_I$.

(a) $\mathbf{A} = \begin{pmatrix}3 & 10\\ -5 & -7\end{pmatrix}, \qquad \mathbf{x}_I = \begin{pmatrix}-3\\2\end{pmatrix}.$

(b) $\mathbf{A} = \begin{pmatrix}8 & -5\\ 5 & -2\end{pmatrix}, \qquad \mathbf{x}_I = \begin{pmatrix}3\\-1\end{pmatrix}.$

(16) Consider the nonlinear planar system

$$x' = y + 1\,,$$
$$y' = 4x - x^2\,.$$

(a) Find all of its equilibrium (critical, stationary) points.
(b) Find a nonconstant function $H(x, y)$ such that every trajectory of the system satisfies $H(x, y) = c$ for some constant c.
(c) Sketch a phase-plane portrait of the system by sketching contour lines of $H(x, y)$. (Carefully mark all sketched trajectories with arrows!)
(d) Classify the type and stability of each equilibrium (critical, stationary) point.

(17) Consider the nonlinear planar system

$$x' = -5y\,,$$
$$y' = x - 4y - x^2\,.$$

(a) Find all of its equilibrium (critical, stationary) points.
(b) Compute the Jacobian matrix (derivative matrix, linearization coefficient matrix) at each equilibrium (critical, stationary) point.
(c) Classify the type and stability of each equilibrium (critical, stationary) point.
(d) Sketch a plausible global phase-plane portrait. (Carefully mark all sketched trajectories with arrows!)

(18) Consider the nonlinear planar system

$$x' = x(3 - 3x + 2y)\,,$$
$$y' = y(6 - x - y)\,.$$

Do parts (a-d) as for the previous problem.
(e) Why do solutions that start in the first quadrant stay in the first quadrant?