Quiz 4 Solutions, Math 246, Professor David Levermore
Tuesday, 1 March 2011

(1) [2] What is the interval of definition for the solution to the initial-value problem

$$\frac{\mathrm{d}^3 z}{\mathrm{d}t^3} + \frac{t^2}{2+t}\, z = \frac{\sin(t)}{t-1}\,, \qquad z(-1) = z'(-1) = z''(-1) = 3\,.$$

Solution. The equation is already in normal form. The coefficient is defined and continuous everywhere except $t = -2$. The forcing is defined and continuous everywhere except $t = 1$. The initial time is $t = -1$. The interval of definition is therefore $(-2, 1)$.

(2) [3] Compute the Wronskian $W[Y_1, Y_2](t)$ of the functions $Y_1(t) = e^t$ and $Y_2(t) = te^t$. (Evaluate the determinant and simplify.)

Solution. Because $Y_1'(t) = e^t$ and $Y_2'(t) = e^t + te^t$, the Wronskian is

$$\begin{aligned} W[Y_1, Y_2](t) &= \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det\begin{pmatrix} e^t & te^t \\ e^t & e^t + te^t \end{pmatrix} \\ &= e^t(e^t + te^t) - e^t(te^t) = e^{2t} + te^{2t} - te^{2t} = e^{2t}\,. \end{aligned}$$

(3) [5] Given that e^t and te^t make up a fundamental set of solutions to $y'' - 2y' + y = 0$, find the solution $Y(t)$ to the general initial-value problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,.$$

Solution. Because e^t and te^t is a fundamental set of solutions to $y'' - 2y' + y = 0$, a general solution has the form $Y(t) = c_1 e^t + c_2 te^t$. Then $Y'(t) = c_1 e^t + c_2(e^t + te^t)$. To satisfy the initial conditions one needs

$$y_0 = Y(0) = c_1\,, \qquad\qquad y_1 = Y'(0) = c_1 + c_2\,.$$

You solve these equations to obtain $c_1 = y_0$ and $c_2 = y_1 - y_0$. The solution to the general initial-value problem is therefore

$$Y(t) = y_0 e^t + (y_1 - y_0)te^t\,.$$

Remark. Because this solution to the general initial-value problem can be written

$$Y(t) = (e^t - te^t)y_0 + te^t y_1\,,$$

the associated natural fundamental set of solutions is given by

$$N_0(t) = e^t - te^t\,, \qquad N_1(t) = te^t\,.$$