Quiz 6 Solutions, Math 246, Professor David Levermore
Tuesday, 15 March 2011

(1) [4] Given that e^t is a particular solution of the differential equation, solve the initial-value problem

$$y'' + 9y = 10e^t\,, \qquad y(0) = 1\,, \quad y'(0) = -5\,.$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its characteristic polynomial is $p(z) = z^2 + 9 = z^2 + 3^2$, which has roots $\pm i3$. A general solution of the associated homogeneous equation is

$$y_H(t) = c_1\cos(3t) + c_2\sin(3t)\,.$$

Because a particular solution is $y_P(t) = e^t$, a general solution of the equation is

$$y(t) = c_1\cos(3t) + c_2\sin(3t) + e^t\,.$$

Because

$$y'(t) = -3c_1\sin(3t) + 3c_2\cos(3t) + e^t\,,$$

the initial conditions imply that

$$1 = y(0) = c_1\cos(0) + c_2\sin(0) + e^0 = c_1 + 1\,,$$
$$-5 = y'(0) = -3c_1\sin(0) + 3c_2\cos(0) + e^0 = 3c_2 + 1\,.$$

Hence, $c_1 = 0$ and $c_2 = -2$. The solution of the initial-value problem is therefore

$$y(t) = -2\sin(3t) + e^t\,.$$

(2) [3] Give the characteristic, degree, and multiplicity for the forcing term of the equation

$$y'' + 4y' + 20y = 7t^3e^{-2t}\sin(4t)\,.$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its characteristic polynomial is $p(z) = z^2 + 4z + 20 = (z+2)^2 + 4^2$, which has roots $-2 \pm i4$.

The forcing term $7t^3e^{-2t}\sin(4t)$ has characteristic $\mu + i\nu = -2 + i4$, degree $d = 3$, and multiplicity $m = 1$.

(3) [3] Give a particular solution of the equation

$$y'' + 4y' + 20y = 5e^{-5t}\,.$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its characteristic polynomial is $p(z) = z^2 + 4z + 20 = (z+2)^2 + 4^2$, which has roots $-2 \pm i4$. Its forcing has characteristic $\mu + i\nu = -5$, degree $d = 0$, and multiplicity $m = 0$.

Key Identity Evaluation. Because $d + m = 0$ you only need the Key identity

$$\mathrm{L}(e^{zt}) = (z^2 + 4z + 20)e^{zt}\,.$$

By evaluating this at the forcing characteristic $z = -5$ you see that

$$\mathrm{L}(e^{-5t}) = \big((-5)^2 + 4\cdot(-5) + 20\big)e^{-5t} = (25 - 20 + 20)e^{-5t} = 25e^{-5t}\,.$$

By dividing both sides of this by 5 you see that a particular solution is $y_P(t) = \frac{1}{5}e^{-5t}$.

Undetermined Coefficients. Because $\mu + i\nu = -5$ and $d = m = 0$, there is a particular solution of the form

$$y_P(t) = Ae^{-5t}\,.$$

Because

$$y_P'(t) = -5Ae^{-5t}\,, \qquad y_P''(t) = 25Ae^{-5t}\,,$$

you find that

$$\begin{aligned} y_P'' + 4y_P' + 20y_P &= 25Ae^{-5t} + 4\cdot(-5Ae^{-5t}) + 20Ae^{-5t} \\ &= (25 - 20 + 20)Ae^{-5t} = 25Ae^{-5t} \\ &= 5e^{-5t}\,. \end{aligned}$$

By setting $25A = 5$ you see that a particular solution is $y_P(t) = \frac{1}{5}e^{-5t}$.

Remark. The fact that forcing term $7t^3e^{-2t}\sin(4t)$ in problem 2 has characteristic $\mu + i\nu = -2 + i4$, degree $d = 3$, and multiplicity $m = 1$ tells you how to start the methods of Key identity evaluations and undetermined coefficients.

- **Key Identity Evaluations:** It tells you to start by evaluating the first ($m = 1$) through fourth ($d + m = 4$) derivatives (with respect to z) of the Key identity at $z = -2 + i4$ ($\mu + i\nu = -2 + i4$). Because $p(z) = z^2 + 4z + 20$, $p'(z) = 2z + 4$, $p''(z) = 2$, and $p'''(z) = 0$, you see that $p'(-2 + i4) = i8$ and $p''(-2 + i4) = 2$. The evaluations of the first through fourth derivatives of the Key identity at $z = -2 + i4$ are

$$\begin{aligned} \mathrm{L}\big(t\,e^{-2t+i4t}\big) &= i8\cdot e^{-2t+i4t}\,, \\ \mathrm{L}\big(t^2e^{-2t+i4t}\big) &= 2\cdot i8\cdot t\,e^{-2t+i4t} + 2\cdot e^{-2t+i4t}\,, \\ \mathrm{L}\big(t^3e^{-2t+i4t}\big) &= 3\cdot i8\cdot t^2\,e^{-2t+i4t} + 3\cdot 2\cdot t\,e^{-2t+i4t}\,, \\ \mathrm{L}\big(t^4e^{-2t+i4t}\big) &= 4\cdot i8\cdot t^3\,e^{-2t+i4t} + 6\cdot 2\cdot t^2e^{-2t+i4t}\,. \end{aligned}$$

- **Undetermined Coefficients:** It tells you to start by seeking a particular solution in the form

$$\begin{aligned} y_P(t) = {} & (A_0t^4 + A_1t^3 + A_2t^2 + A_3t)e^{-2t}\cos(4t) \\ & + (B_0t^4 + B_1t^3 + B_2t^2 + B_3t)e^{-2t}\sin(4t)\,. \end{aligned}$$

  Here the powers of t run from fourth ($d + m = 4$) to first ($m = 1$) while $e^{-2t}\cos(4t)$ and $e^{-2t}\sin(4t)$ are because $\mu + i\nu = -2 + i4$.