Quiz 7 Solutions, Math 246, Professor David Levermore
Tuesday, 29 March 2011

(1) [4] Compute the Green function associated with the differential operator $\mathrm{L} = \mathrm{D}^2 - 4$.

Solution. The Green function $g(t)$ satisfies the initial-value problem

$$g'' - 4g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

The characteristic polynomial is $p(z) = z^2 + 16$, which has roots ± 2. Therefore $g(t)$ has the form

$$g(t) = c_1 e^{2t} + c_2 e^{-2t}\,.$$

Because $g'(t) = 2c_1 e^{2t} - 2c_2 e^{-2t}$, the initial conditions imply

$$0 = g(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2\,, \qquad 1 = g'(0) = 2c_1 e^0 - 2c_2 e^0 = 2c_1 - 2c_2\,.$$

The solution of this system is $c_1 = \frac{1}{4}$, $c_2 = -\frac{1}{4}$. Therefore the Green function $g(t)$ is

$$g(t) = \tfrac{1}{4}\big(e^{2t} - e^{-2t}\big)\,.$$

(2) [2] The displacement of a spring-mass system is given by

$$h(t) = 5\cos(3t) - 12\sin(3t)\,.$$

What is the amplitude of this simple harmonic motion?

Solution. The amplitude-phase form of the displacement is

$$h(t) = A\cos(3t - \delta) = A\cos(\delta)\cos(3t) + A\sin(\delta)\sin(3t)\,.$$

Comparing this with the given form shows that $A\cos(\delta) = 5$ and $A\sin(\delta) = -12$. The applitude A is therefore given by

$$A = \sqrt{5^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13\,.$$

Remark. The phase δ is given by $\delta = 2\pi - \tan^{-1}(\frac{12}{5})$.

(3) [4] A sping-mass system is governed by the initial-value problem

$$h'' + \gamma h' + 36h = 0\,, \qquad h(0) = 2\,, \quad h'(0) = -1\,.$$

(a) Determine the natural frequency and period of the spring.
(b) For what value of γ is the system critically damped?

Solution (a). Because the equation is in normal form, the natural frequency ω_o is

$$\omega_o = \sqrt{36} = 6\,.$$

The natural period is therefore $T_o = 2\pi/\omega_o = \frac{1}{3}\pi$.

Solution (b). The characteristic polynomial is

$$p(z) = z^2 + \gamma z + 36 = (z + \tfrac{1}{2}\gamma)^2 + 36 - \tfrac{1}{4}\gamma^2\,.$$

The system will be critically damped when $36 = \frac{1}{4}\gamma^2$, which is when

$$\gamma = \sqrt{4 \cdot 36} = 2 \cdot 6 = 12\,.$$