Quiz 10, Solutions Math 246, Professor David Levermore
Tuesday, 26 April 2011

(1) [4] Let $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}$. Compute $e^{t\mathbf{A}}$.

Solution. The characteristic polynomial is $p(z) = z^2 - 6z + 9 = (z-3)^2$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^{3t}\big[\mathbf{I} + t\,(\mathbf{A} - 3\mathbf{I})\big] \\ &= e^{3t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}\right] \\ &= e^{3t}\begin{pmatrix} 1+2t & 2t \\ -2t & 1-2t \end{pmatrix}. \end{aligned}$$

(2) [3] $\mathbf{A} = \begin{pmatrix} 6 & 3 \\ 3 & -2 \end{pmatrix}$ has eigenvalues -3 and 7. Find an eigenvector for each eigenvalue.

Solution. One has $\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{A} - 7\mathbf{I} = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}$.
The eigenvectors $\mathbf{v}_1$ associated with the eigenvalue -3 satisfy $(\mathbf{A} + 3\mathbf{I})\mathbf{v}_1 = 0$. You can either solve this system or simply read-off from a nonzero column of $\mathbf{A} - 7\mathbf{I}$ that these have the form

$$\mathbf{v}_1 = \alpha_1 \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad \text{for some } \alpha_1 \neq 0\,.$$

The eigenvectors $\mathbf{v}_2$ associated with the eigenvalue 7 satisfy $(\mathbf{A} - 7\mathbf{I})\mathbf{v}_2 = 0$. You can either solve this system or simply read-off from a nonzero column of $\mathbf{A} + 3\mathbf{I}$ that these have the form

$$\mathbf{v}_2 = \alpha_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{for some } \alpha_2 \neq 0\,.$$

(3) [3] A real 2×2 matrix $\mathbf{A}$ has the eigenpair $\left(2 - i\,,\, \begin{pmatrix} 3 \\ -i \end{pmatrix}\right)$. Diagonalize $\mathbf{A}$.

Solution: Because $\mathbf{A}$ is real, a second eigenpair will be the complex conjugate of the given eigenpair. If you use the eigenpairs

$$\left(2 - i\,,\, \begin{pmatrix} 3 \\ -i \end{pmatrix}\right), \qquad \left(2 + i\,,\, \begin{pmatrix} 3 \\ i \end{pmatrix}\right),$$

then set

$$\mathbf{V} = \begin{pmatrix} 3 & 3 \\ -i & i \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 2-i & 0 \\ 0 & 2+i \end{pmatrix}.$$

Because $\det(\mathbf{V}) = 3 \cdot i - (-i) \cdot 3 = 6i$, you obtain the diagonalization

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \begin{pmatrix} 3 & 3 \\ -i & i \end{pmatrix}\begin{pmatrix} 2-i & 0 \\ 0 & 2+i \end{pmatrix}\frac{1}{6i}\begin{pmatrix} i & -3 \\ i & 3 \end{pmatrix}.$$