Quiz 11 Solutions, Math 246, Professor David Levermore
Tuesday, 10 May 2011

(1) [5] Consider the first-order planar system

$$x' = x(3-y)\,, \qquad y' = y(x-2)\,.$$

(a) Find all of its stationary points.
(b) Find a nonconstant function $H(x,y)$ such that every trajectory of this system that lies in the first quadrant satisfies $H(x,y) = c$ for some constant c.

Solution (a). The stationary points satisfy

$$0 = x(3-y)\,, \qquad 0 = y(x-2)\,.$$

The first equation is satisfies if and only if $x = 0$ or $y = 3$, while the second is satisfies if and only if $x = 2$ or $y = 0$. All the stationary points are therefore $(0,0)$ and $(2,3)$.

Solution (b). The trajectory equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(x-2)}{x(3-y)}\,.$$

This equation is separable. Its solutions satisfy

$$\int \frac{x-2}{x}\,\mathrm{d}x = \int \frac{3-y}{y}\,\mathrm{d}y\,,$$

which for (x,y) in the first quadrant integrates to

$$x - 2\log(x) = 3\log(y) - y + c\,.$$

Therefore you can set $H(x,y) = x - 2\log(x) + y - 3\log(y)$.

(2) [5] Consider the first-order planar system

$$x' = y\,, \qquad y' = -5x + 5x^2 - 4y\,.$$

Its stationary points are $(0,0)$ and $(1,0)$. Classify the type and stability of each of these stationary points. (You do not have to sketch anything.)

Solution. The matrix of partial derivatives is $\begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -5+10x & -4 \end{pmatrix}$.

- At $(0,0)$ the coefficient matrix of its linearization is $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix}$, which has the characteristic polynomial $p(z) = z^2 + 4z + 5 = (z+2)^2 + 1$. The eigenvalues of $\mathbf{A}$ are the conjugate pair $-2 \pm i$. Because these have negative real part, and because $a_{21} = -5 < 0$, the stationary point $(-5,0)$ is a *clockwise spiral sink* and is therefore *attracting*.
- At $(1,0)$ the coefficient matrix of its linearization is $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 5 & -4 \end{pmatrix}$, which has the characteristic polynomial $p(z) = z^2 + 4z - 5 = (z+5)(z-1)$. The eigenvalues of $\mathbf{A}$ are -5 and 1. Because these are real, nonzero, and have opposite sign, the stationary point $(0,0)$ is a *saddle* and is therefore *unstable*.