Large Deviations and Importance Sampling for Weakly Interacting Diffusions*

Zack Bezemek

Boston University
Advisor: Konstantinos Spiliopoulos

February 28, 2023

* Joint work with Max Heldman. Z.B. partially supported by NSF-DMS 1550918.

Overview

Setup and Motivation: Weakly Interacting Diffusions

We consider particles $\{X^{i,N}\}_{i=1}^{N}$ obeying the system of N SDEs:

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N}, \mu_t^N)dW_t^i,\ X_0^{i,N} = x^{i,N},$$

Setup and Motivation: Weakly Interacting Diffusions

We consider particles $\{X^{i,N}\}_{i=1}^N$ obeying the system of N SDEs:

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N}, \mu_t^N)dW_t^i,\ X_0^{i,N} = x^{i,N},$$

where $\{W^i\}_{i=1}^N$ are independent m-dimensional standard Brownian motions,
$b : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d, \sigma : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times m}$, and

Setup and Motivation: Weakly Interacting Diffusions

We consider particles $\{X^{i,N}\}_{i=1}^N$ obeying the system of N SDEs:

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N}, \mu_t^N)dW_t^i, \; X_0^{i,N} = x^{i,N},$$

where $\{W^i\}_{i=1}^N$ are independent m-dimensional standard Brownian motions,
$b : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d, \sigma : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times m}$, and

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}, \quad t \in [0, T], \quad \text{"Empirical Measure"}.$$

Setup and Motivation: Weakly Interacting Diffusions

We consider particles $\{X^{i,N}\}_{i=1}^N$ obeying the system of N SDEs:

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N}, \mu_t^N)dW_t^i,\ X_0^{i,N} = x^{i,N},$$

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}, \quad t \in [0, T], \quad \text{"Empirical Measure"}$$

Applications:

- mathematical biology and neuroscience (Delarue et al., 2015)
- machine learning (V. Nilsson and P. Nyquist, 2022)
- stochastic portfolio theory (E. R. Fernholz, 2002)
- systemic risk/clustering of defaults (J. Garnier et al., 2013; P. Dai Pra et al., 2009)
- many more (Chaintron and Diez, 2022)

Propagation of Chaos

Theorem: Propagation of Chaos (M. Kac, 1957; H.P. McKean, 1969)

Let

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N}, \mu_t^N)dW_t^i, \; X_0^{i,N} = x^{i,N} \qquad (1)$$

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}}.$$

Then μ^N converges weakly as $N \to \infty$ as a $C([0,T];\mathcal{P}(\mathbb{R}^d))$ -valued random variable to the deterministic process $\mathcal{L}(X)$, where X_t satisfies:

$$dX_t = b(X_t, \mathcal{L}(X_t))dt + \sigma(X_t, \mathcal{L}(X_t))dW_t, \quad X_0 \sim \nu \qquad (2)$$

$$\nu := \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{x^{i,N}}.$$

Propagation of Chaos

Theorem: Propagation of Chaos (M. Kac, 1957; H.P. McKean, 1969)

Let

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N}, \mu_t^N)dW_t^i, \; X_0^{i,N} = x^{i,N} \tag{1}$$

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}}.$$

Then μ^N converges weakly as $N \to \infty$ as a $C([0,T];\mathcal{P}(\mathbb{R}^d))$ -valued random variable to the deterministic process $\mathcal{L}(X)$, where X_t satisfies:

$$dX_t = b(X_t, \mathcal{L}(X_t))dt + \sigma(X_t, \mathcal{L}(X_t))dW_t, \quad X_0 \sim \nu \tag{2}$$

$$\nu := \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{x^{i,N}}.$$

Further suppose $x^{i,N}$ are IID with law ν. Then as $N \to \infty$, any k "tagged" particles obeying (1) converge to independent processes obeying Equation (2).

The Classical Model of Dawson

(D.A. Dawson, 1983) considered the one-dimensional system:

$$dX_t^{i,N} = \left[-(X_t^{i,N})^3 + X_t^{i,N} - \theta\left(X_t^{i,N} - \frac{1}{N}\sum_{j=1}^{N} X_t^{j,N}\right)\right]dt + \sigma dW_t^i$$

$i \in \{1, ..., N\}, \theta, \sigma > 0$. This corresponds to $b(x, \mu) = -x^3 + x - \theta(x - \int z\mu(dz))$.

The Classical Model of Dawson

(D.A. Dawson, 1983) considered the one-dimensional system:

$$dX_t^{i,N} = \left[-(X_t^{i,N})^3 + X_t^{i,N} - \theta \left(X_t^{i,N} - \frac{1}{N} \sum_{j=1}^{N} X_t^{j,N} \right) \right] dt + \sigma dW_t^i$$

$i \in \{1, ..., N\}, \theta, \sigma > 0$. This corresponds to $b(x, \mu) = -x^3 + x - \theta(x - \int z\mu(dz))$. It was shown:

- The system undergoes the propagation of chaos

The Classical Model of Dawson

Setup and Motivation

Importance Sampling

Small-Noise Diffusions

Weakly Interacting Diffusions

Numerical Examples

(D.A. Dawson, 1983) considered the one-dimensional system:

$$dX_t^{i,N} = \left[-(X_t^{i,N})^3 + X_t^{i,N} - \theta \left(X_t^{i,N} - \frac{1}{N} \sum_{j=1}^{N} X_t^{j,N} \right) \right] dt + \sigma dW_t^i$$

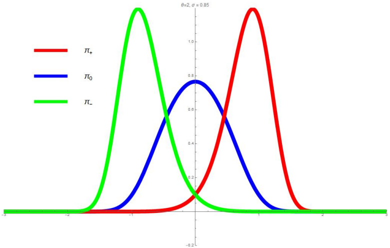
$i \in \{1, ..., N\}, \theta, \sigma > 0$. This corresponds to $b(x, \mu) = -x^3 + x - \theta(x - \int z\mu(dz))$. It was shown:

- The system undergoes the propagation of chaos
- There exists a critical temperature $\sigma_c > 0$ such that:

Temperature (σ)	Finite N Particles	$N \to \infty$ Limit
$\sigma \geq \sigma_c$	$\exists!$ invariant density	$\exists!$ invariant density
$0 < \sigma < \sigma_c$	$\exists!$ invariant density	3 invariant densities

The Classical Model of Dawson

$$dX_t = \left[-(X_t)^3 + X_t - \theta \Big(X_t - \mathbb{E}[X_t] \Big) \right] dt + \sigma dW_t$$



Suppose $\mu_0^N \to \pi_+$. Then for $T > 0$ and $\pi_+ \notin A \subset \mathcal{P}(\mathbb{R}^d)$:

$$\mathbb{P}(\mu_T^N \in A) \to 0 \text{ as } N \to \infty.$$

Setup and Motivation

Importance Sampling

Small-Noise Diffusions

Weakly Interacting Diffusions

Numerical Examples

The Classical Model of Dawson

Setup and Motivation

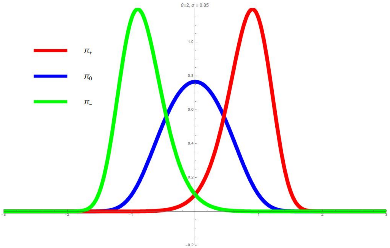
Importance Sampling

Small-Noise Diffusions

Weakly Interacting Diffusions

Numerical Examples

$$dX_t = \left[-(X_t)^3 + X_t - \theta \Big(X_t - \mathbb{E}[X_t] \Big) \right] dt + \sigma dW_t$$



Suppose $\mu_0^N \to \pi_+$. Then for $T > 0$ and $\pi_+ \notin A \subset \mathcal{P}(\mathbb{R}^d)$:

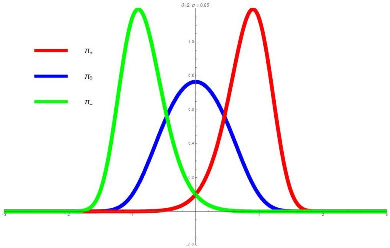
$$\mathbb{P}(\mu_T^N \in A) \to 0 \text{ as } N \to \infty.$$

For example:

$$\mathbb{P}\left(\frac{1}{N} \sum_{i=1}^{N} X_T^{i,N} < 0 \right) \to 0 \text{ as } N \to \infty.$$

The Classical Model of Dawson

$$dX_t = \left[-(X_t)^3 + X_t - \theta\Big(X_t - \mathbb{E}[X_t]\Big)\right]dt + \sigma dW_t$$



Suppose $\mu_0^N \to \pi_+$. Then for $T > 0$ and $\pi_+ \notin A \subset \mathcal{P}(\mathbb{R}^d)$:

$$\mathbb{P}(\mu_T^N \in A) \to 0 \text{ as } N \to \infty.$$

For example:

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} X_T^{i,N} < 0\right) \to 0 \text{ as } N \to \infty.$$

Our long-term goal: Efficiently estimate such a quantity for large N.

The Problem with Standard Monte Carlo (1)

- We will simplify the problem to estimating $\mathbb{E}[\exp(-NG(X^N))]$ for $G \in C_b(\mathcal{X})$ and $\{X^N\} \subset \mathcal{X}$.

The Problem with Standard Monte Carlo (1)

- We will simplify the problem to estimating $\mathbb{E}[\exp(-NG(X^N))]$ for $G \in C_b(\mathcal{X})$ and $\{X^N\} \subset \mathcal{X}$.
- Corresponds to $\mathbb{P}(X^N \in A)$ by taking $G(x) = 0$ for $x \in A$ and $G(x) = +\infty$ otherwise (P. Dupuis et al., 2012).

The Problem with Standard Monte Carlo (1)

- We will simplify the problem to estimating $\mathbb{E}[\exp(-NG(X^N))]$ for $G \in C_b(\mathcal{X})$ and $\{X^N\} \subset \mathcal{X}$.
- Corresponds to $\mathbb{P}(X^N \in A)$ by taking $G(x) = 0$ for $x \in A$ and $G(x) = +\infty$ otherwise (P. Dupuis et al., 2012).
- Let $\hat{X}^N$ have Law $\hat{\mathbb{P}}^N$ and X^N have Law $\mathbb{P}^N$. Consider the unbiased estimator:

$$\hat{\delta}(N) = \frac{1}{M}\sum_{j=1}^{M} \exp(-NG(\hat{X}^{N,j}))\frac{d\mathbb{P}^N}{d\hat{\mathbb{P}}^N}(\hat{X}^{N,j}),$$

where $\{\hat{X}^{N,j}\}_{j=1}^{M}$ are M independent samples of $\hat{X}^N$.

The Problem with Standard Monte Carlo (1)

- We will simplify the problem to estimating $\mathbb{E}[\exp(-NG(X^N))]$ for $G \in C_b(\mathcal{X})$ and $\{X^N\} \subset \mathcal{X}$.
- Corresponds to $\mathbb{P}(X^N \in A)$ by taking $G(x) = 0$ for $x \in A$ and $G(x) = +\infty$ otherwise (P. Dupuis et al., 2012).
- Let $\hat{X}^N$ have Law $\hat{\mathbb{P}}^N$ and X^N have Law $\mathbb{P}^N$. Consider the unbiased estimator:

$$\hat{\delta}(N) = \frac{1}{M}\sum_{j=1}^{M} \exp(-NG(\hat{X}^{N,j}))\frac{d\mathbb{P}^N}{d\hat{\mathbb{P}}^N}(\hat{X}^{N,j}),$$

where $\{\hat{X}^{N,j}\}_{j=1}^{M}$ are M independent samples of $\hat{X}^N$.
- Measure performance via relative error:

$$\rho(\hat{\delta}(N)) = \frac{\sqrt{\mathrm{Var}(\hat{\delta}(N))}}{\mathbb{E}[\hat{\delta}(N)]} = \frac{1}{\sqrt{M}}\sqrt{R(\hat{\delta}(N)) - 1}$$

$$R(\hat{\delta}(N)) = \mathbb{E}\left[\exp(-2NG(\hat{X}^N))\left(\frac{d\mathbb{P}^N}{d\hat{\mathbb{P}}^N}(\hat{X}^N)\right)^2\right]/\mathbb{E}[\exp(-NG(X^N))]^2$$

The Problem with Standard Monte Carlo (2)

Definition

We say that a sequence of $\mathcal{X}$-valued random variables $\{X^N\}_{N\in\mathbb{N}}$ satisfies the *Laplace Principle* (equivalently LDP) with rate function $I:\mathcal{X}\to[0,+\infty]$ if for all $G\in C_b(\mathcal{X})$,

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\left[\exp\left(-NG(X^N)\right)\right]=-\inf_{x\in\mathcal{X}}\left[G(x)+I(x)\right].$$

The Problem with Standard Monte Carlo (2)

Definition

We say that a sequence of $\mathcal{X}$-valued random variables $\{X^N\}_{N\in\mathbb{N}}$ satisfies the *Laplace Principle* (equivalently LDP) with rate function $I : \mathcal{X} \to [0, +\infty]$ if for all $G \in C_b(\mathcal{X})$,

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left[\exp\left(-NG(X^N)\right)\right] = -\inf_{x\in\mathcal{X}}\left[G(x) + I(x)\right].$$

Suppose $\{X^N\}$ satisfies the LDP with rate function I. Then:

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[\exp(-2NG(X^N))] = -\inf_{x\in\mathcal{X}}\left[2G(x) + I(x)\right] := -\gamma_2,$$

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[\exp(-NG(X^N))] = -\inf_{x\in\mathcal{X}}\left[G(x) + I(x)\right] := -\gamma_1.$$

The Problem with Standard Monte Carlo (2)

Definition

We say that a sequence of $\mathcal{X}$-valued random variables $\{X^N\}_{N\in\mathbb{N}}$ satisfies the *Laplace Principle* (equivalently LDP) with rate function $I:\mathcal{X}\to[0,+\infty]$ if for all $G\in C_b(\mathcal{X})$,

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\left[\exp\left(-NG(X^N)\right)\right]=-\inf_{x\in\mathcal{X}}\left[G(x)+I(x)\right].$$

Suppose $\{X^N\}$ satisfies the LDP with rate function I. Then:

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}[\exp(-2NG(X^N))]=-\inf_{x\in\mathcal{X}}\left[2G(x)+I(x)\right]:=-\gamma_2,$$

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}[\exp(-NG(X^N))]=-\inf_{x\in\mathcal{X}}\left[G(x)+I(x)\right]:=-\gamma_1.$$

Letting $\delta(N)$ be the standard M.C. Estimator ($\hat{X}^N=X^N$):

$$\rho(\delta(N))=\frac{1}{\sqrt{M}}\sqrt{R(\delta(N))-1},\quad R(\delta(N))=\frac{\mathbb{E}[\exp(-2NG(X^N))]}{\mathbb{E}[\exp(-NG(X^N))]^2}$$

$$\lim_{N\to\infty}\frac{1}{N}\log R(\delta(N))=2\gamma_1-\gamma_2\geq 0$$

The Problem with Standard Monte Carlo (2)

Definition

We say that a sequence of $\mathcal{X}$-valued random variables $\{X^N\}_{N\in\mathbb{N}}$ satisfies the *Laplace Principle* (equivalently LDP) with rate function $I : \mathcal{X} \to [0, +\infty]$ if for all $G \in C_b(\mathcal{X})$,

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left[\exp\left(-NG(X^N)\right)\right] = -\inf_{x\in\mathcal{X}}\left[G(x) + I(x)\right].$$

Suppose $\{X^N\}$ satisfies the LDP with rate function I. Then:

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[\exp(-2NG(X^N))] = -\inf_{x\in\mathcal{X}}\left[2G(x) + I(x)\right] := -\gamma_2,$$

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[\exp(-NG(X^N))] = -\inf_{x\in\mathcal{X}}\left[G(x) + I(x)\right] := -\gamma_1.$$

Letting $\delta(N)$ be the standard M.C. Estimator ($\hat{X}^N = X^N$):

$$\rho(\delta(N)) = \frac{1}{\sqrt{M}}\sqrt{R(\delta(N)) - 1}, \quad R(\delta(N)) = \frac{\mathbb{E}[\exp(-2NG(X^N))]}{\mathbb{E}[\exp(-NG(X^N))]^2}$$

$$\lim_{N\to\infty} \frac{1}{N} \log R(\delta(N)) = 2\gamma_1 - \gamma_2 \geq 0$$

- So Standard M.C. has exponentially poor performance in N.

Log Efficiency

$$\rho(\hat{\delta}(N)) = \frac{1}{\sqrt{M}}\sqrt{R(\hat{\delta}(N)) - 1}$$

$$R(\hat{\delta}(N)) = \frac{\mathbb{E}\left[\exp(-2NG(\hat{X}^N))\left(\frac{d\mathbb{P}^N}{d\hat{\mathbb{P}}^N}(\hat{X}^N)\right)^2\right]}{\mathbb{E}[\exp(-NG(X^N))]^2}$$

Definition: Log-Efficient

$\hat{\delta}(N)$ is called log-efficient if

$$\lim_{N\to\infty} -\frac{1}{N}\log R(\hat{\delta}(N)) = 0.$$

Example: Small-Noise Diffusions (1)

- Let $\mathcal{X} = C([0,T];\mathbb{R}^d)$, $G(\phi) = g(\phi(T))$, and X^N be given by

$$dX_t^N = b(X_t^N)dt + \frac{1}{\sqrt{N}}\sigma(X_t^N)dW_t.$$

Example: Small-Noise Diffusions (1)

- Let $\mathcal{X} = C([0,T];\mathbb{R}^d)$, $G(\phi) = g(\phi(T))$, and X^N be given by
$$dX_t^N = b(X_t^N)dt + \frac{1}{\sqrt{N}}\sigma(X_t^N)dW_t.$$

- A natural change of measure is given by Girsanov's Theorem, so
$$d\hat{X}_t^{N,v} = [b(\hat{X}_t^{N,v}) + \sigma(\hat{X}_t^{N,v})v(t,\hat{X}_t^{N,v})]dt + \frac{1}{\sqrt{N}}\sigma(\hat{X}_t^{N,v})dW_t$$
$$\frac{d\mathbb{P}^N}{d\hat{\mathbb{P}}^{N,v}}(\hat{X}^{N,v}) = \exp\left(-\sqrt{N}\int_0^T v(t,\hat{X}_t^{N,v})\cdot dW_t - \frac{N}{2}\int_0^T |v(t,\hat{X}_t^{N,v})|^2 dt\right)$$
for some $v : [0,T]\times\mathbb{R}^d \to \mathbb{R}^m$.

Example: Small-Noise Diffusions (1)

- Let $\mathcal{X} = C([0,T];\mathbb{R}^d)$, $G(\phi) = g(\phi(T))$, and X^N be given by
$$dX_t^N = b(X_t^N)dt + \frac{1}{\sqrt{N}}\sigma(X_t^N)dW_t.$$
- A natural change of measure is given by Girsanov's Theorem, so
$$d\hat{X}_t^{N,v} = [b(\hat{X}_t^{N,v}) + \sigma(\hat{X}_t^{N,v})v(t,\hat{X}_t^{N,v})]dt + \frac{1}{\sqrt{N}}\sigma(\hat{X}_t^{N,v})dW_t$$
$$\frac{d\mathbb{P}^N}{d\hat{\mathbb{P}}^{N,v}}(\hat{X}^{N,v}) = \exp\left(-\sqrt{N}\int_0^T v(t,\hat{X}_t^{N,v})\cdot dW_t - \frac{N}{2}\int_0^T |v(t,\hat{X}_t^{N,v})|^2 dt\right)$$
for some $v : [0,T]\times\mathbb{R}^d \to \mathbb{R}^m$.
- Letting $v^N = -\sigma^\top\nabla_x\Phi^N$, where:
$$-\partial_t\Phi^N - b\cdot\nabla_x\Phi^N + \frac{1}{2}|\sigma^\top\nabla_x\Phi^N|^2 - \frac{1}{2N}\sigma\sigma^\top : \nabla_x^2\Phi^N = 0$$
$$\Phi^N(T,x) = g(x),$$
we have v^N provides a zero-variance estimator.

Example: Small-Noise Diffusions (1)

- Let $\mathcal{X} = C([0,T];\mathbb{R}^d)$, $G(\phi) = g(\phi(T))$, and X^N be given by

$$dX_t^N = b(X_t^N)dt + \frac{1}{\sqrt{N}}\sigma(X_t^N)dW_t.$$

- A natural change of measure is given by Girsanov's Theorem, so

$$d\hat{X}_t^{N,v} = [b(\hat{X}_t^{N,v}) + \sigma(\hat{X}_t^{N,v})v(t,\hat{X}_t^{N,v})]dt + \frac{1}{\sqrt{N}}\sigma(\hat{X}_t^{N,v})dW_t$$

$$\frac{d\mathbb{P}^N}{d\hat{\mathbb{P}}^{N,v}}(\hat{X}^{N,v}) = \exp\left(-\sqrt{N}\int_0^T v(t,\hat{X}_t^{N,v})\cdot dW_t - \frac{N}{2}\int_0^T |v(t,\hat{X}_t^{N,v})|^2 dt\right)$$

for some $v : [0,T]\times\mathbb{R}^d \to \mathbb{R}^m$.

- Letting $v^N = -\sigma^\top \nabla_x \Phi^N$, where:

$$-\partial_t\Phi^N - b\cdot\nabla_x\Phi^N + \frac{1}{2}|\sigma^\top\nabla_x\Phi^N|^2 - \frac{1}{2N}\sigma\sigma^\top : \nabla_x^2\Phi^N = 0$$

$$\Phi^N(T,x) = g(x),$$

we have v^N provides a zero-variance estimator.

- Sending $N\to\infty$, we expect that $v = -\sigma^\top\nabla_x\Psi$ for Ψ solving the associated zero-viscocity HJB Equation will yield small relative error uniformly in N.

Example: Small-Noise Diffusions (2)

$$dX_t^N = b(X_t^N)dt + \frac{1}{\sqrt{N}}\sigma(X_t^N)dW_t$$

$$d\hat{X}_t^{N,v} = [b(\hat{X}_t^{N,v}) + \sigma(\hat{X}_t^{N,v})v(t,\hat{X}_t^{N,v})]dt + \frac{1}{\sqrt{N}}\sigma(\hat{X}_t^{N,v})dW_t$$

$$-\partial_t\Psi - b\cdot\nabla_x\Psi + \frac{1}{2}|\sigma^\top\nabla_x\Psi|^2 = 0, \quad \Psi(T,x) = g(x).$$

Theorem (E. Vanden-Eijnden and J. Weare, 2012)

The importance sampling scheme for estimating $\mathbb{E}[\exp(-Ng(X_T^N))]$ using $v(t,x) = -\sigma^\top(x)\nabla_x\Psi(t,x)$ is log-efficient.

Example: Small-Noise Diffusions (2)

$$dX_t^N = b(X_t^N)dt + \frac{1}{\sqrt{N}}\sigma(X_t^N)dW_t$$

$$d\hat{X}_t^{N,v} = [b(\hat{X}_t^{N,v}) + \sigma(\hat{X}_t^{N,v})v(t,\hat{X}_t^{N,v})]dt + \frac{1}{\sqrt{N}}\sigma(\hat{X}_t^{N,v})dW_t$$

$$-\partial_t\Psi - b\cdot\nabla_x\Psi + \frac{1}{2}|\sigma^\top\nabla_x\Psi|^2 = 0, \quad \Psi(T,x) = g(x).$$

Theorem (E. Vanden-Eijnden and J. Weare, 2012)

The importance sampling scheme for estimating $\mathbb{E}[\exp(-Ng(X_T^N))]$ using $v(t,x) = -\sigma^\top(x)\nabla_x\Psi(t,x)$ is log-efficient. Further:

$$\Psi(s,x) = \inf_{\phi\in C([s,T];\mathbb{R}^d):\phi(s)=x}\left\{J^s(\phi) + g(\phi(T))\right\} \tag{3}$$

where $J^s : C([s,T];\mathbb{R}^d) \to [0,+\infty]$ is the LDP rate function for $\{X^N\}$ starting at position x at time s:

$$J^s(\phi) = \frac{1}{2}\int_s^T |(\dot{\phi}(t) - b(\phi(t)))^\top[\sigma\sigma^\top(\phi(t))]^{-1}(\dot{\phi}(t) - b(\phi(t)))|dt$$

$$= \inf_{u:[s,T]\to\mathbb{R}^m:\dot{\phi}(t)=b(\phi(t))+\sigma(\phi(t))u(t)} \frac{1}{2}\int_s^T |u(t)|^2 dt.$$

Back to Interacting Diffusions (1)

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N}, \mu_t^N)dW_t^i, \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$$

- Want to estimate $\mathbb{E}[\exp(-Ng(\mu_T^N))]$ for $g : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$.
- Examples: $g(\mu) = \int_{\mathbb{R}^d} f(z)\mu(dz)$, $g(\mu) = \int_{\mathbb{R}^d} f(z-y)\mu(dz)\mu(dy)$, ect.

Back to Interacting Diffusions (1)

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N}, \mu_t^N)dW_t^i, \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$$

- Want to estimate $\mathbb{E}[\exp(-Ng(\mu_T^N))]$ for $g : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$.
- Examples: $g(\mu) = \int_{\mathbb{R}^d} f(z)\mu(dz)$, $g(\mu) = \int_{\mathbb{R}^d} f(z-y)\mu(dz)\mu(dy)$, ect.

Use again Girsanov:

$$d\hat{X}_t^{i,N,v} = [b(\hat{X}_t^{i,N,v}, \hat{\mu}_t^{N,v}) + \sigma(\hat{X}_t^{i,N,v}, \hat{\mu}_t^{N,v})v_i(t, \hat{X}_t^{1,N,v}, ..., \hat{X}_t^{N,N,v})]dt$$

$$+ \sigma(\hat{X}_t^{i,N,v}, \hat{\mu}_t^{N,v})dW_t^i, \quad \hat{\mu}_t^{N,v} := \frac{1}{N}\sum_{i=1}^N \delta_{\hat{X}_t^{i,N,v}}$$

$$\frac{\mathbb{P}^N}{\hat{\mathbb{P}}^{N,v}}(\hat{\mu}^{N,v}) = \exp\Big(-\sum_{i=1}^N \int_0^T v_i(t, \hat{X}_t^{1,N,v}, ..., \hat{X}_t^{N,N,v}) \cdot dW_t^i$$

$$+ \frac{1}{2}\int_0^T |v_i(t, \hat{X}_t^{1,N,v}, ..., \hat{X}_t^{N,N,v})|^2 dt\Big)$$

For $v : [0, T] \times \mathbb{R}^{dN} \to \mathbb{R}^{mN}$.

Back to Interacting Diffusions (2)

$v_i^N(t, x_1, \ldots, x_N) = -N\sigma^\top(x_i, \mu_x^N)\nabla_{x_i}\tilde{\Phi}^N(t, x_1, \ldots, x_N)$ yields a zero-variance estimator, where $\mu_x^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ and:

$$-\partial_t \tilde{\Phi}^N(t,x) = \sum_{i=1}^N \left\{ b(x_i, \mu_x^N) \cdot \nabla_{x_i}\tilde{\Phi}^N(t,x) - \frac{N}{2}\left|\sigma^\top(x_i, \mu_x^N)\nabla_{x_i}\tilde{\Phi}^N(t,x)\right|^2 + \frac{1}{2}\sigma\sigma^\top(x_i, \mu_x^N) : \nabla_{x_i}^2 \tilde{\Phi}^N(t,x) \right\}, x \in \mathbb{R}^{Nd}, t \in [0, T)$$

$$\tilde{\Phi}^N(T, x_1, \ldots, x_N) = g(\mu_x^N).$$

Back to Interacting Diffusions (2)

$v_i^N(t, x_1, \dots, x_N) = -N\sigma^\top(x_i, \mu_x^N)\nabla_{x_i}\tilde{\Phi}^N(t, x_1, \dots, x_N)$ yields a zero-variance estimator, where $\mu_x^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ and:

$$-\partial_t\tilde{\Phi}^N(t,x) = \sum_{i=1}^N \Big\{ b(x_i, \mu_x^N)\cdot\nabla_{x_i}\tilde{\Phi}^N(t,x) - \frac{N}{2}|\sigma^\top(x_i,\mu_x^N)\nabla_{x_i}\tilde{\Phi}^N(t,x)|^2 + \frac{1}{2}\sigma\sigma^\top(x_i,\mu_x^N) : \nabla_{x_i}^2\tilde{\Phi}^N(t,x)\Big\}, x \in \mathbb{R}^{Nd}, t \in [0, T)$$

$\tilde{\Phi}^N(T, x_1, \dots, x_N) = g(\mu_x^N)$.

- Question: How can we send $N \to \infty$ to get a "zero-viscocity" HJB Equation?

Back to Interacting Diffusions (2)

$v_i^N(t, x_1, \dots, x_N) = -N\sigma^\top(x_i, \mu_x^N)\nabla_{x_i}\tilde{\Phi}^N(t, x_1, \dots, x_N)$ yields a zero-variance estimator, where $\mu_x^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ and:

$$-\partial_t\tilde{\Phi}^N(t,x) = \sum_{i=1}^N \Big\{ b(x_i, \mu_x^N)\cdot\nabla_{x_i}\tilde{\Phi}^N(t,x) - \frac{N}{2}|\sigma^\top(x_i,\mu_x^N)\nabla_{x_i}\tilde{\Phi}^N(t,x)|^2 + \frac{1}{2}\sigma\sigma^\top(x_i,\mu_x^N) : \nabla_{x_i}^2\tilde{\Phi}^N(t,x)\Big\}, x \in \mathbb{R}^{Nd}, t \in [0,T)$$

$\tilde{\Phi}^N(T, x_1, \dots, x_N) = g(\mu_x^N)$.

- Question: How can we send $N \to \infty$ to get a "zero-viscocity" HJB Equation?
- Let $\Phi^N : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be such that $\Phi^N(\mu_x^N) = \tilde{\Phi}^N(x_1, \dots, x_N)$.

Back to Interacting Diffusions (2)

$v_i^N(t, x_1, ..., x_N) = -N\sigma^\top(x_i, \mu_x^N)\nabla_{x_i}\tilde{\Phi}^N(t, x_1, ..., x_N)$ yields a zero-variance estimator, where $\mu_x^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ and:

$$-\partial_t\tilde{\Phi}^N(t,x) = \sum_{i=1}^N \Big\{ b(x_i, \mu_x^N)\cdot\nabla_{x_i}\tilde{\Phi}^N(t,x) - \frac{N}{2}|\sigma^\top(x_i,\mu_x^N)\nabla_{x_i}\tilde{\Phi}^N(t,x)|^2 + \frac{1}{2}\sigma\sigma^\top(x_i,\mu_x^N) : \nabla^2_{x_i}\tilde{\Phi}^N(t,x)\Big\}, x\in\mathbb{R}^{Nd}, t\in[0,T)$$

$\tilde{\Phi}^N(T, x_1, ..., x_N) = g(\mu_x^N)$.

- Question: How can we send $N \to \infty$ to get a "zero-viscocity" HJB Equation?
- Let $\Phi^N : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be such that $\Phi^N(\mu_x^N) = \tilde{\Phi}^N(x_1, ..., x_N)$.
- Then by Propositions 5.35 and 5.91 of (R. Carmona and F. Delarue, 2018):

$$\nabla_{x_i}\tilde{\Phi}^N(x) = \frac{1}{N}\partial_\mu\Phi^N(\mu_x^N)[x_i]$$

$$\nabla^2_{x_i}\tilde{\Phi}^N(x) = \frac{1}{N}\partial_z\partial_\mu\Phi^N(\mu_x^N)[x_i] + \frac{1}{N^2}\partial^2_\mu\Phi^N(\mu_x^N)[x_i, x_i]$$

- $\partial_\mu\Phi^N(\nu)[\cdot]$ denotes the **Lions derivative** (P. L. Lions 2012)

Back to Interacting Diffusions (3)

$$v_i^N(t, x_1, ..., x_N) = -\sigma^\top(x_i, \mu_x^N)\partial_\mu \Phi^N(t, \mu_x^N)[x_i]$$

Back to Interacting Diffusions (3)

$$v_i^N(t, x_1, \dots, x_N) = -\sigma^\top(x_i, \mu_x^N)\partial_\mu \Phi^N(t, \mu_x^N)[x_i]$$

Plugging into the Equation $\tilde{\Phi}^N$ solves, $\Phi^N(T, \mu_x^N) = g(\mu_x^N)$, and:

$$-\partial_t \Phi^N(t, \mu_x^N) =$$

$$\frac{1}{N}\sum_{i=1}^{N}\Big\{ b(x_i, \mu_x^N)\cdot \partial_\mu \Phi^N(t, \mu_x^N)[x_i] - \frac{1}{2}|\sigma^\top(x_i, \mu_x^N)\partial_\mu \Phi^N(t, \mu_x^N)[x_i]|^2$$

$$+\frac{1}{2}\sigma\sigma^\top(x_i, \mu_x^N) : [\partial_z \partial_\mu \Phi^N(t, \mu_x^N)[x_i] + \frac{1}{N}\partial_\mu^2 \Phi^N(t, \mu_x^N)[x_i, x_i]]\Big\}, x \in \mathbb{R}^{Nd}.$$

Back to Interacting Diffusions (3)

$$v_i^N(t, x_1, ..., x_N) = -\sigma^\top(x_i, \mu_x^N)\partial_\mu \Phi^N(t, \mu_x^N)[x_i]$$

Plugging into the Equation $\tilde{\Phi}^N$ solves, $\Phi^N(T, \mu_x^N) = g(\mu_x^N)$, and:

$$-\partial_t \Phi^N(t, \mu_x^N) =$$

$$\frac{1}{N}\sum_{i=1}^N \Big\{ b(x_i, \mu_x^N) \cdot \partial_\mu \Phi^N(t, \mu_x^N)[x_i] - \frac{1}{2}|\sigma^\top(x_i, \mu_x^N)\partial_\mu \Phi^N(t, \mu_x^N)[x_i]|^2$$

$$+ \frac{1}{2}\sigma\sigma^\top(x_i, \mu_x^N) : [\partial_z\partial_\mu \Phi^N(t, \mu_x^N)[x_i] + \frac{1}{N}\partial_\mu^2 \Phi^N(t, \mu_x^N)[x_i, x_i]] \Big\}, x \in \mathbb{R}^{Nd}.$$

We rewrite the sum as an integral, so:

$$-\partial_t \Phi^N(t, \mu_x^N) =$$

$$\int_{\mathbb{R}^d} b(z, \mu_x^N) \cdot \partial_\mu \Phi^N(t, \mu_x^N)[z] - \frac{1}{2}|\sigma^\top(z, \mu_x^N)\partial_\mu \Phi^N(t, \mu_x^N)[z]|^2$$

$$+ \frac{1}{2}\sigma\sigma^\top(z, \mu_x^N) : [\partial_z\partial_\mu \Phi^N(t, \mu_x^N)[z] + \frac{1}{N}\partial_\mu^2 \Phi^N(t, \mu_x^N)[z, z]]\mu_x^N(dz), x \in \mathbb{R}^{Nd}.$$

Back to Interacting Diffusions (3)

$$v_i^N(t, x_1, ..., x_N) = -\sigma^\top(x_i, \mu_x^N)\partial_\mu \Phi^N(t, \mu_x^N)[x_i]$$

Plugging into the Equation $\tilde{\Phi}^N$ solves, $\Phi^N(T, \mu_x^N) = g(\mu_x^N)$, and:

$$-\partial_t \Phi^N(t, \mu_x^N) =$$

$$\frac{1}{N}\sum_{i=1}^N \Big\{ b(x_i, \mu_x^N) \cdot \partial_\mu \Phi^N(t, \mu_x^N)[x_i] - \frac{1}{2}|\sigma^\top(x_i, \mu_x^N)\partial_\mu \Phi^N(t, \mu_x^N)[x_i]|^2$$

$$+ \frac{1}{2}\sigma\sigma^\top(x_i, \mu_x^N) : [\partial_z \partial_\mu \Phi^N(t, \mu_x^N)[x_i] + \frac{1}{N}\partial_\mu^2 \Phi^N(t, \mu_x^N)[x_i, x_i]] \Big\}, x \in \mathbb{R}^{Nd}.$$

We rewrite the sum as an integral, so:

$$-\partial_t \Phi^N(t, \mu_x^N) =$$

$$\int_{\mathbb{R}^d} b(z, \mu_x^N) \cdot \partial_\mu \Phi^N(t, \mu_x^N)[z] - \frac{1}{2}|\sigma^\top(z, \mu_x^N)\partial_\mu \Phi^N(t, \mu_x^N)[z]|^2$$

$$+ \frac{1}{2}\sigma\sigma^\top(z, \mu_x^N) : [\partial_z \partial_\mu \Phi^N(t, \mu_x^N)[z] + \frac{1}{N}\partial_\mu^2 \Phi^N(t, \mu_x^N)[z, z]]\mu_x^N(dz), x \in \mathbb{R}^{Nd}.$$

Now we have all terms only depend on the measure $\mu_x^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$, and we can send $N \to \infty$!

Main Theorem

$$-\partial_t \Psi(t,\nu) = \int_{\mathbb{R}^d} b(z,\nu)\cdot \partial_\mu \Psi(t,\nu)[z] - \frac{1}{2}\left|\sigma^\top(z,\nu)\partial_\mu \Psi(t,\nu)[z]\right|^2 + \frac{1}{2}\sigma\sigma^\top(z,\nu) : \partial_z \partial_\mu \Psi(t,\nu)[z]\nu(dz),\ \ \Psi(T,\nu) = g(\nu), \nu \in \mathcal{P}_2(\mathbb{R}^d)$$

Main Theorem

$$-\partial_t \Psi(t,\nu) = \int_{\mathbb{R}^d} b(z,\nu)\cdot\partial_\mu\Psi(t,\nu)[z] - \frac{1}{2}|\sigma^\top(z,\nu)\partial_\mu\Psi(t,\nu)[z]|^2 + \frac{1}{2}\sigma\sigma^\top(z,\nu) : \partial_z\partial_\mu\Psi(t,\nu)[z]\nu(dz),\ \ \Psi(T,\nu) = g(\nu), \nu\in\mathcal{P}_2(\mathbb{R}^d)$$

$$dX_t^{i,N} = b(X_t^{i,N},\mu_t^N)dt + \sigma(X_t^{i,N},\mu_t^N)dW_t^i, \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$$

$$d\hat{X}_t^{i,N,v} = [b(\hat{X}_t^{i,N,v},\hat{\mu}_t^{N,v}) + \sigma(\hat{X}_t^{i,N,v},\hat{\mu}_t^{N,v})v(t,\hat{\mu}_t^{N,v},\hat{X}_t^{i,N,v})]dt + \sigma(\hat{X}_t^{i,N,v},\hat{\mu}_t^{N,v})dW_t^i, \quad \hat{\mu}_t^{N,v} := \frac{1}{N}\sum_{i=1}^N \delta_{\hat{X}_t^{i,N,v}}$$

Main Theorem

$$-\partial_t \Psi(t,\nu) = \int_{\mathbb{R}^d} b(z,\nu)\cdot\partial_\mu\Psi(t,\nu)[z] - \frac{1}{2}|\sigma^\top(z,\nu)\partial_\mu\Psi(t,\nu)[z]|^2 + \frac{1}{2}\sigma\sigma^\top(z,\nu) : \partial_z\partial_\mu\Psi(t,\nu)[z]\nu(dz),\ \ \Psi(T,\nu) = g(\nu), \nu\in\mathcal{P}_2(\mathbb{R}^d)$$

$$dX_t^{i,N} = b(X_t^{i,N},\mu_t^N)dt + \sigma(X_t^{i,N},\mu_t^N)dW_t^i, \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$$

$$d\hat{X}_t^{i,N,v} = [b(\hat{X}_t^{i,N,v},\hat{\mu}_t^{N,v}) + \sigma(\hat{X}_t^{i,N,v},\hat{\mu}_t^{N,v})v(t,\hat{\mu}_t^{N,v},\hat{X}_t^{i,N,v})]dt + \sigma(\hat{X}_t^{i,N,v},\hat{\mu}_t^{N,v})dW_t^i, \quad \hat{\mu}_t^{N,v} := \frac{1}{N}\sum_{i=1}^N \delta_{\hat{X}_t^{i,N,v}}$$

Theorem (Z. B. and M. Heldman, 2022)

The IS scheme for estimating $\mathbb{E}[\exp(-Ng(\mu_T^N))]$ using $v(t,\nu,z) = -\sigma^\top(z,\nu)\partial_\mu\Psi(t,\nu)[z]$ is log-efficient.

Main Theorem

$$-\partial_t \Psi(t,\nu) = \int_{\mathbb{R}^d} b(z,\nu)\cdot \partial_\mu \Psi(t,\nu)[z] - \frac{1}{2}|\sigma^\top(z,\nu)\partial_\mu\Psi(t,\nu)[z]|^2 + \frac{1}{2}\sigma\sigma^\top(z,\nu) : \partial_z\partial_\mu\Psi(t,\nu)[z]\nu(dz),\ \Psi(T,\nu) = g(\nu), \nu\in\mathcal{P}_2(\mathbb{R}^d)$$

$$dX_t^{i,N} = b(X_t^{i,N},\mu_t^N)dt + \sigma(X_t^{i,N},\mu_t^N)dW_t^i, \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$$

$$d\hat{X}_t^{i,N,v} = [b(\hat{X}_t^{i,N,v},\hat{\mu}_t^{N,v}) + \sigma(\hat{X}_t^{i,N,v},\hat{\mu}_t^{N,v})v(t,\hat{\mu}_t^{N,v},\hat{X}_t^{i,N,v})]dt + \sigma(\hat{X}_t^{i,N,v},\hat{\mu}_t^{N,v})dW_t^i, \quad \hat{\mu}_t^{N,v} := \frac{1}{N}\sum_{i=1}^N \delta_{\hat{X}_t^{i,N,v}}$$

Theorem (Z. B. and M. Heldman, 2022)

The IS scheme for estimating $\mathbb{E}[\exp(-Ng(\mu_T^N))]$ using $v(t,\nu,z) = -\sigma^\top(z,\nu)\partial_\mu\Psi(t,\nu)[z]$ is log-efficient. Futher:

$$\Psi(s,\nu) = \inf_{\mu\in C([s,T];\mathcal{P}(\mathbb{R}^d)):\mu(s)=\nu} \{I^s(\mu) + g(\mu_T)\}$$

where $I^s : C([s,T];\mathcal{P}(\mathbb{R}^d)) \to [0,+\infty]$ is the LDP rate function for μ^N with initial distribution $\mu_s^N \to \nu$.

Main Theorem

$$
\begin{aligned}
v(t,\nu,z) &= -\sigma^\top(z,\nu)\partial_\mu \Psi(t,\nu)[z] \\
\Psi(s,\nu) &= \inf_{\mu\in C([s,T];\mathbb{R}^d):\mu(s)=\nu} \{I^s(\mu)+g(\mu_T)\} \\
I^s(\mu) &= \frac{1}{2}\int_s^T \sup_{\phi\in C_c^\infty(\mathbb{R}^d):\text{den}\neq 0} \frac{|\langle \dot\mu(t)-L^*_{\mu(t)}\mu(t),\phi\rangle|^2}{\langle \mu(t), \left\|\nabla^\top\phi\sigma\sigma^\top\nabla\phi\right\|^2\rangle} dt \\
L_\nu\phi(x) &= b(x,\nu)\cdot\nabla\phi(x)+\frac{1}{2}\sigma\sigma^\top(x,\nu):\nabla^2\phi(x) \\
I^s(\mu) &= \inf_{u\in\mathcal{U}:\mu(t)=\mathcal{L}(\hat X_t^u),t\in[s,T]} \frac{1}{2}\mathbb{E}\left[\int_s^T |u(t)|^2 dt\right] \\
d\hat X_t^u &= [b(\hat X_t^u,\mathcal{L}(\hat X_t^u))+\sigma(\hat X_t^u,\mathcal{L}(\hat X_t^u))u(t)]dt+\sigma(\hat X_t^u,\mathcal{L}(\hat X_t^u))dW_t, \quad \hat X_s^u\sim\nu
\end{aligned}
$$

- Rate function representations from (D. A. Dawson and J. Gärtner, 1987) and (A. Budhiraja et al., 2012)

Main Theorem

$$v(t,\nu,z) = -\sigma^\top(z,\nu)\partial_\mu \Psi(t,\nu)[z]$$

$$\Psi(s,\nu) = \inf_{\mu \in C([s,T];\mathbb{R}^d):\mu(s)=\nu} \{I^s(\mu) + g(\mu_T)\}$$

$$I^s(\mu) = \frac{1}{2}\int_s^T \sup_{\phi \in C_c^\infty(\mathbb{R}^d):\text{den}\neq 0} \frac{|\langle \dot{\mu}(t) - L^*_{\mu(t)}\mu(t), \phi\rangle|^2}{\langle \mu(t), \left\|\nabla^\top \phi \sigma\sigma^\top \nabla\phi\right\|^2\rangle} dt$$

$$L_\nu \phi(x) = b(x,\nu)\cdot\nabla\phi(x) + \frac{1}{2}\sigma\sigma^\top(x,\nu) : \nabla^2\phi(x)$$

$$I^s(\mu) = \inf_{u \in \mathcal{U}:\mu(t)=\mathcal{L}(\hat{X}^u_t), t\in[s,T]} \frac{1}{2}\mathbb{E}\left[\int_s^T |u(t)|^2 dt\right]$$

$$d\hat{X}^u_t = [b(\hat{X}^u_t, \mathcal{L}(\hat{X}^u_t)) + \sigma(\hat{X}^u_t, \mathcal{L}(\hat{X}^u_t))u(t)]dt + \sigma(\hat{X}^u_t, \mathcal{L}(\hat{X}^u_t))dW_t, \quad \hat{X}^u_s \sim \nu$$

- Rate function representations from (D. A. Dawson and J. Gärtner, 1987) and (A. Budhiraja et al., 2012)
- Assuming sufficient regularity for an expansion of the prelimit HJB equation, can further prove vanishing relative error in N

About the HJB Equation on Wasserstein Space

$$
\begin{aligned}
-\partial_t \Psi(t,\nu) = \int_{\mathbb{R}^d} b(z,\nu)\cdot \partial_\mu \Psi(t,\nu)[z] - \frac{1}{2}|\sigma^\top(z,\nu)\partial_\mu \Psi(t,\nu)[z]|^2 \\
+ \frac{1}{2}\sigma\sigma^\top(z,\nu) : \partial_z \partial_\mu \Psi(t,\nu)[z]\nu(dz) \\
\Psi(T,\nu) = g(\nu)
\end{aligned}
$$

- Similar to the master equation in mean-field games (J.F. Chassagneux et al., 2022; R. Carmona and F. Delarue, 2014)

About the HJB Equation on Wasserstein Space

$$
\begin{aligned}
-\partial_t \Psi(t,\nu) &= \int_{\mathbb{R}^d} b(z,\nu)\cdot\partial_\mu\Psi(t,\nu)[z] - \frac{1}{2}|\sigma^\top(z,\nu)\partial_\mu\Psi(t,\nu)[z]|^2 \\
&\quad + \frac{1}{2}\sigma\sigma^\top(z,\nu) : \partial_z\partial_\mu\Psi(t,\nu)[z]\nu(dz) \\
\Psi(T,\nu) &= g(\nu)
\end{aligned}
$$

- Similar to the master equation in mean-field games (J.F. Chassagneux et al., 2022; R. Carmona and F. Delarue, 2014)
- Subsolutions are sufficient for log-efficiency. Uniqueness/regularity is active area (C. Wu and J. Zhang, 2020; A. Cosso et al., 2021; H.M. Soner and Q. Yan, 2022). Zero-viscosity limit studied in (M. Germain et al., 2021).

About the HJB Equation on Wasserstein Space

$$-\partial_t \Psi(t,\nu) = \int_{\mathbb{R}^d} b(z,\nu)\cdot\partial_\mu\Psi(t,\nu)[z] - \frac{1}{2}|\sigma^\top(z,\nu)\partial_\mu\Psi(t,\nu)[z]|^2 + \frac{1}{2}\sigma\sigma^\top(z,\nu) : \partial_z\partial_\mu\Psi(t,\nu)[z]\nu(dz)$$

$$\Psi(T,\nu) = g(\nu)$$

- Similar to the master equation in mean-field games (J.F. Chassagneux et al., 2022; R. Carmona and F. Delarue, 2014)
- Subsolutions are sufficient for log-efficiency. Uniqueness/regularity is active area (C. Wu and J. Zhang, 2020; A. Cosso et al., 2021; H.M. Soner and Q. Yan, 2022). Zero-viscosity limit studied in (M. Germain et al., 2021).
- Related equation used to prove quantitative rates of propagation of chaos in (P. E. Chaudru de Raynal and N. Frikha, 2021; J.F. Chassagneux et al., 2022)

About the HJB Equation on Wasserstein Space

$$
\begin{aligned}
-\partial_t \Psi(t,\nu) &= \int_{\mathbb{R}^d} b(z,\nu)\cdot \partial_\mu \Psi(t,\nu)[z] - \frac{1}{2}|\sigma^\top(z,\nu)\partial_\mu \Psi(t,\nu)[z]|^2 \\
&\quad + \frac{1}{2}\sigma\sigma^\top(z,\nu) : \partial_z \partial_\mu \Psi(t,\nu)[z]\nu(dz) \\
\Psi(T,\nu) &= g(\nu)
\end{aligned}
$$

- Similar to the master equation in mean-field games (J.F. Chassagneux et al., 2022; R. Carmona and F. Delarue, 2014)
- Subsolutions are sufficient for log-efficiency. Uniqueness/regularity is active area (C. Wu and J. Zhang, 2020; A. Cosso et al., 2021; H.M. Soner and Q. Yan, 2022). Zero-viscosity limit studied in (M. Germain et al., 2021).
- Related equation used to prove quantitative rates of propagation of chaos in (P. E. Chaudru de Raynal and N. Frikha, 2021; J.F. Chassagneux et al., 2022)
- Numerical methods for solving for the optimal control (J.P. Fouque and Z. Zhang, 2020; M. Laurière, 2021)

Numerical Example 1

$$dX_t^{i,N,y} = [-X_t^{i,N,y} + \frac{2}{N}\sum_{j=1}^{N} X_t^{j,N,y}]dt + \sigma dW_t^i, X_0^{i,N,y} = y$$

$$g(\nu) = \int_{\mathbb{R}} z^2 \nu(dz), \quad \text{target} = \mathbb{E}\left[\exp\left(-\sum_{i=1}^{N}(X_T^{i,N,y})^2\right)\right]$$

$$v(t,\nu,x) = -2\sigma\left[\frac{e^{2t}}{e^{2T}(1+\sigma^2)-e^{2t}\sigma^2}x + \left(\frac{e^{2T}}{e^{2T}\sigma^2 - e^{2t}(\sigma^2-1)} - \frac{e^{2t}}{e^{2T}(1+\sigma^2)-e^{2t}\sigma^2}\right)\int z\nu(dz)\right]$$

Example ($\sigma = .5$, $T = 1$, $y = .2$)

N	IS Est.	IS Rel.Err.	MC Est.	MC Rel.Err.	Exact
5	$2.3816 \cdot 10^{-1}$	$3.4601 \cdot 10^{-2}$	$2.3807 \cdot 10^{-1}$	1.0380	$2.3747 \cdot 10^{-1}$
10	$8.2550 \cdot 10^{-2}$	$2.7101 \cdot 10^{-2}$	$8.2551 \cdot 10^{-2}$	1.5721	$8.2412 \cdot 10^{-2}$
15	$2.8641 \cdot 10^{-2}$	$2.4001 \cdot 10^{-2}$	$2.8661 \cdot 10^{-2}$	2.1957	$2.8600 \cdot 10^{-2}$
20	$9.9373 \cdot 10^{-3}$	$2.2369 \cdot 10^{-2}$	$9.9165 \cdot 10^{-3}$	2.9589	$9.9254 \cdot 10^{-3}$
25	$3.4486 \cdot 10^{-3}$	$2.1310 \cdot 10^{-2}$	$3.4824 \cdot 10^{-3}$	3.9086	$3.4445 \cdot 10^{-3}$
30	$1.1968 \cdot 10^{-3}$	$2.0550 \cdot 10^{-2}$	$1.1951 \cdot 10^{-1}$	5.1415	$1.1954 \cdot 10^{-3}$
50	$1.7361 \cdot 10^{-5}$	$1.8934 \cdot 10^{-2}$	$1.7569 \cdot 10^{-5}$	14.5202	$1.7339 \cdot 10^{-5}$
80	$3.0339 \cdot 10^{-8}$	$1.8003 \cdot 10^{-2}$	$3.2426 \cdot 10^{-8}$	74.1871	$3.0289 \cdot 10^{-8}$

Numerical Example 1

$$dX_t^{i,N,y} = [-X_t^{i,N,y} + \frac{2}{N}\sum_{j=1}^{N} X_t^{j,N,y}]dt + \sigma dW_t^i, X_0^{i,N,y} = y$$

$$g(\nu) = \int_{\mathbb{R}} z^2 \nu(dz), \quad \text{target} = \mathbb{E}\left[\exp\left(-\sum_{i=1}^{N}(X_T^{i,N,y})^2\right)\right]$$

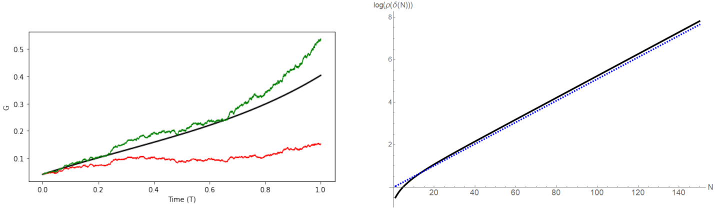


Figure: Left: A realization of $G(\mu_T^N)$ (green) and $G(\hat{\mu}_T^N)$ (red) against the second moment of the limiting McKean-Vlasov Equation (black) for $T \in [0, 1]$ and $N = 50$. Right: The log-scale exact standard MC relative error as N increases (black).

Numerical Example 2

$$dX_t^{i,N,y} = [-X_t^{i,N,y} + \frac{2}{N}\sum_{j=1}^{N} X_t^{j,N,y}]dt + \sigma dW_t^i, X_0^{i,N,y} = y$$

$$g(\nu) = \left|\int_{\mathbb{R}} z\nu(dz)\right|, \quad \text{target} = \mathbb{E}\left[\exp\left(-\left|\sum_{i=1}^{N} X_T^{i,N,y}\right|\right)\right]$$

$$v(t,\nu,x) = -\sigma e^{T-t}\text{sign}\left(\int z\nu(dz)\right)$$

Example ($\sigma = .5$, $T = 1$, $y = .4$)

N	IS Est.	IS Rel.Err.	MC Est.	MC Rel.Err.
5	$2.6644 \cdot 10^{-2}$	$3.7070 \cdot 10^{-1}$	$2.6653 \cdot 10^{-2}$	2.9930
10	$9.0804 \cdot 10^{-4}$	$3.1427 \cdot 10^{-1}$	$9.2009 \cdot 10^{-4}$	12.8032
15	$3.0269 \cdot 10^{-5}$	$2.6350 \cdot 10^{-1}$	$3.0645 \cdot 10^{-5}$	53.9410
20	$9.9533 \cdot 10^{-7}$	$2.2227 \cdot 10^{-1}$	$1.0625 \cdot 10^{-6}$	337.0401
25	$3.2471 \cdot 10^{-8}$	$1.8900 \cdot 10^{-1}$	$3.5791 \cdot 10^{-8}$	561.9164
30	$1.0537 \cdot 10^{-9}$	$1.6154 \cdot 10^{-1}$	$1.2253 \cdot 10^{-9}$	1016.1656

Importance Sampling vs. Standard Monte Carlo Estimate and Relative Error

Numerical Example 3

$$dX_t^{i,N,y} = [-X_t^{i,N,y} + \frac{2}{N}\sum_{j=1}^{N} X_t^{j,N,y}]dt + \sigma dW_t^i, X_0^{i,N,y} = y$$

$$g(\nu) = \int_{\mathbb{R}} |z|\nu(dz), \quad \text{target} = \mathbb{E}\left[\exp\left(-\sum_{i=1}^{N}|X_T^{i,N,y}|\right)\right]$$

$$v(t,\nu,x) = -\sigma\left[e^{t-T}\text{sign}(x) + \left(e^{T-t} - e^{t-T}\right)\int \text{sign}(z)\nu(dz)\right]$$

Example ($\sigma = .5$, $T = 1$, $y = .4$)

N	IS Est.	IS Rel.Err.	MC Est.	MC Rel.Err.
5	$2.1355 \cdot 10^{-2}$	$4.7870 \cdot 10^{-1}$	$2.1382 \cdot 10^{-2}$	2.3263
10	$5.6703 \cdot 10^{-4}$	$6.0605 \cdot 10^{-1}$	$5.6806 \cdot 10^{-4}$	6.6374
15	$1.5087 \cdot 10^{-5}$	$7.2452 \cdot 10^{-1}$	$1.5040 \cdot 10^{-5}$	17.3337
20	$4.0157 \cdot 10^{-7}$	$8.3416 \cdot 10^{-1}$	$4.1260 \cdot 10^{-7}$	51.1058
25	$1.0687 \cdot 10^{-8}$	$9.4327 \cdot 10^{-1}$	$1.0296 \cdot 10^{-8}$	97.8820
30	$2.8470 \cdot 10^{-10}$	1.0524	$2.8800 \cdot 10^{-10}$	280.5945

Importance Sampling vs. Standard Monte Carlo Estimate and Relative Error

Extensions and Future Work

Future directions:

- Solve for the optimal controls on the fly
- Extend analysis to estimate probabilities
- Importance sampling scheme for long-time/metastable behavior (P. Dupuis et al., 2015)
- Moderate deviations based importance sampling (Z. B. and K. Spiliopoulos, 2022a; I. Gasteratos et al., 2022; F. Delarue and A. Tse, 2021)
- Joint small noise and large N limit (A. Budhiraja and M. Conroy, 2022)

Gratitude

Thank you to my advisor, Kostas Spiliopoulos, to my coauthor and friend Max Heldman, and to all of you for listening!

Questions?

References I

S. Adams, N. Dirr, M. A. Peletier, and J. Zimmer. 2013. *Large deviations and gradient flows*, Philosophical Transactions of the Royal Society A **371**, no. 2005.

J. Barré, C. Bernardin, R. Chétrite, Y. Chopra, and M. Mariani. 2019. *Gamma Convergence Approach For The Large Deviations Of The Density In Systems Of Interacting Diffusion Processes*, arXiv: 1910.04026.

Bashiri, K. 2020. *On the long-time behaviour of McKean-Vlasov paths*, Electronic Communications in Probability **25**.

K. Bashiri and G. Menz. 2019. *Metastability in a continuous mean-field model at low temperature and strong interaction*, arXiv:1910.11828.

Z. B. and M. Heldman. 2022. *Importance Sampling for the Empirical Measure of Weakly Interacting Diffusions*, arXiv:2210.08591.

Z. B. and K. Spiliopoulos. 2023. *Large deviations for interacting multiscale particle systems*, Stochastic Processes and their Applications **115**, 27–108.

References II

———. 2022a. *Moderate deviations for fully coupled multiscale weakly interacting particle systems*, arXiv:2202.08403.

———. 2022b. *Rate of homogenization for fully-coupled McKean-Vlasov SDEs*, arXiv:2202.07753, to appear in Stochastics and Dynamics.

J. Binney and S. Tremaine. 2008. *Galactic Dynamics*, Princeton University Press, Princeton.

M. Boué and P. Dupuis. 1998. *A variational representation for certain functionals of Brownian motion*, Ann. Prob. **26(4)**, 1641-1659.

A. Budhiraja and M. Conroy. 2022. *Empirical Measure and Small Noise Asymptotics Under Large Deviation Scaling for Interacting Diffusions*, Journal of Theoretical Probability **35**, 295–349.

A. Budhiraja, P. Dupuis, and M. Fischer. 2012. *Large Devation Properties of Weakly Interacting Particles via Weak Convergence Methods*, T.A. of Prob. **40**, 74-100.

References III

P. Cardaliaguet. 2013. *Notes on mean field games (from P. L. Lions' lectures at Collège de France)*, `https://www.ceremade.dauphine.fr/~cardaliaguet/MFG20130420.pdf`, unpublished.

R. Carmona and F. Delarue. 2018. *Probabilistic Theory of Mean Field Games with Applications I*, Springer, NY.

________. 2014. *The Master Equation for Large Population Equilibriums*, Stochastic Analysis and Applications 2014.

Chaintron, L.P. and A. Diez. 2022. *Propagation of chaos: A review of models, methods and applications. ii. applications*, Kinetic and Related Models **15**, no. 6, 1017–1173.

J.F. Chassagneux, D. Crisan, and F. Delarue. 2022. *A Probabilistic Approach to Classical Solutions of the Master Equation for Large Population Equilibria*, Memoirs of the American Mathematical Society **280**.

References IV

J.F. Chassagneux, L. Szpruch, and A. Tse. 2022. *Weak quantitative propagation of chaos via differential calculus on the space of measures*, Ann. Appl. Probab. **32**, no. 3, 1929-1969.

P. E. Chaudru de Raynal and N. Frikha. 2021. *From the backward Kolmogorov PDE on the Wasserstein space to propagation of chaos for McKean-Vlasov SDEs*, Journal de Mathématiques Pures et Appliqués **156**, no. 2.

A. Cosso, F. Gozzi, I. Kharroubi, H. Pham, and M. Rosestolato. 2021. *Master Bellman equation in the Wasserstein space: Uniqueness of viscosity solutions*, arXiv:2107.10535.

P. Dai Pra, W.J. Runggaldier, Elena Sartori, and M. Tolotti. 2009. *Large Portfolio Losses: A Dynamic Contagion Model*, The Annals of Applied Probability **19**, no. 1.

D.A. Dawson. 1983. *Critical dynamics and fluctuations for a mean-field model of cooperative behavior*, J. Stat. Phys. **31**, 29-85.

References V

D. A. Dawson and J. Gärtner. 1986. *Large deviations and tunnelling for particle systems with mean field interaction*, Comptes Rendus Mathématiques de l'Académie des Sciences **8**.

______. 1987. *Large deviations from the mckean-vlasov limit for weakly interacting diffusions*, Stochastics **20(4)**, 247-308.

______. 1988. *Long time behaviour of interacting diffusions*, Stochastic calculus in application.

Delarue, F., J. Inglis, S. Rubenthaler, and E. Tanré. 2015. *Particle systems with a singular mean-field self-excitation. application to neuronal networks*, Stochastic Processes and their Applications **125**, no. 6, 2451–2492.

F. Delarue and A. Tse. 2021. *Uniform in time weak propagation of chaos on the torus*, arXiv:2104.14973.

P. Dupuis and R. S. Ellis. 1997. *A Weak Convergence Approach to the Theory of Large Deviations*, Wiley, NY.

References VI

References

Large Deviations

Differentiation on the Space of Measures

P. Dupuis, K. Spiliopoulos, and X. Zhou. 2015. *Escaping from an attractor: Importance sampling and rest points I*, Ann. Appl. Probab. **25**, no. 5, 2909–2958.

P. Dupuis, K. Spiliopoulos, and H. Wang. 2012. *Importance Sampling for Multiscale Diffusions*, Multiscale Modeling & Simulation **10**, no. 1, 1-27.

P. Dupuis and H. Wang. 2004. *Importance Sampling, Large Deviations, and Differential Games*, Stochastics and Stochastic Reports **76**, no. 6, 481–508.

______. 2007. *Subsolutions of an Isaacs Equation and Efficient Schemes for Importance Sampling*, Mathematics of Operations Research **32**, no. 3, 723–757.

P. Dupuis and K. Spiliopoulos. 2012. *Large deviations for multiscale diffusion via weak convergence methods*, Stochastic Processes and their Applications **122(4)**, 1947-1987.

References VII

P. Dupuis, K. Spiliopoulos, and H. Wang. 2012. *Importance Sampling for Multiscale Diffusions*, SIAM Journal on Multiscale Modeling and Simulation **12**, 1-27.

F. Farkhooi and W. Stannat. 2017. *Complete Mean-Field Theory for Dynamics of Binary Recurrent Networks*, Physical Review Letters **119**.

J. Feng and G. Kurtz. 2006. *Large Deviations for Stochastic Processes*, American Mathematical Society, Providence.

E. R. Fernholz. 2002. *Stochastic portfolio theory*, Springer, NY.

M. I. Freidlin and A. D. Wentzell. 2012. *Random Perturbations of Dynamical Systems*, Springer, Heidelberg.

M. Fischer. 2014. *On the form of the large deviation rate function for the empirical measures of weakly interacting systems*, Bernoulli **20(4)**, 1765-1801.

J.P. Fouque and Z. Zhang. 2020. *Deep Learning Methods for Mean Field Control Problems With Delay*, Frontiers in Applied Mathematics and Statistics **6**, no. 11.

References VIII

J. Garnier, G. Papanicolaou, and T. Yang. 2013. *Large Deviations for a Mean Field Model of Systemic Risk*, SIAM Journal on Financial Mathematics **4(1)**, 151-184.

I. Gasteratos, M. Salins, and K. Spiliopoulos. 2022. *Importance sampling for stochastic reaction-diffusion equations in the moderate deviation regime*, arXiv:2206.00646.

M. Germain, H. Pham, and X. Warin. 2021. *Rate of convergence for particle approximation of PDEs in Wasserstein space*, arXiv:2103.00837.

M. Kac. 1957. *Probability and Related Topics in Physical Sciences*, Amer Mathematical Society, Providence.

D. Lacker. 2016. *Limit Theory for Controlled McKean–Vlasov Dynamics*, SIAM Journal on Control and Optimization **55(3)**.

M. Laurière. 2021. *Numerical Methods for Mean Field Games and Mean Field Type Control*, arXiv:2106.06231.

S. Lifson and J. Jackson. 1961. *On the Self-Diffusion of Ions in a Polyelectrolyte Solution*, J. Chemical Physics **36**, 2410-2414.

References IX

M. Mariani. 2012. *A Gamma-convergence approach to large deviations*, arXiv: 1204.0640.

H.P. McKean. 1969. *Propagation of chaos for a class of non-linear parabolic equations*, Lecture Series in Differential Equations, Volume 2 **19**, 177–194.

V. Nilsson and P. Nyquist. 2022. *A note on large deviations for interacting particle dynamics for finding mixed equilibria in zero-sum games*, arXiv:2206.15177.

G. dos Reis, G. Smith, and P. Tankov. 2018. *Importance sampling for McKean-Vlasov SDEss*, arXiv:1803.09320.

H.M. Soner and Q. Yan. 2022. *Viscosity Solutions for McKean-Vlasov Control on a torus*, arXiv:2212.11053.

K. Spiliopoulos. 2015. *Nonasymptotic Performance Analysis of Importance Sampling Schemes for Small Noise Diffusions*, Journal of Applied Probability **53**, no. 3, 797-–810.

References X

E. Vanden-Eijnden and J. Weare. 2012. *Rare Event Simulation of Small Noise Diffusions*, Communications on Pure and Applied Mathematics **65**, no. 12, 1770-–1803.

C. Wu and J. Zhang. 2020. *Viscosity solutions to parabolic master equations and McKean-Vlasov SDEs with closed-loop controls*, Annals of Applied Probability **30**, no. 2, 936–986.

Introduction to Large Deviations

Definition

Let $\{X^N\}_{N\in\mathbb{N}}$ be a sequence of random variables taking values in a complete separable metric space $\mathcal{X}$. We say that $\{X^N\}$ satisfies the *large deviation principle* with rate function I if there exists $I : \mathcal{X} \to [0, \infty]$ with compact level sets such that

1. $\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(X^N \in F) \leq -\inf_{x\in F} I(x)$, $\forall F \subset \mathcal{X}$ such that F is closed
2. $\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}(X^N \in G) \geq -\inf_{x\in G} I(x)$, $\forall G \subset \mathcal{X}$ such that G is open.

Introduction to Large Deviations

Definition

Let $\{X^N\}_{N\in\mathbb{N}}$ be a sequence of random variables taking values in a complete separable metric space $\mathcal{X}$. We say that $\{X^N\}$ satisfies the *large deviation principle* with rate function I if there exists $I : \mathcal{X} \to [0, \infty]$ with compact level sets such that

1. $\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(X^N \in F) \leq -\inf_{x\in F} I(x)$, $\forall F \subset \mathcal{X}$ such that F is closed
2. $\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}(X^N \in G) \geq -\inf_{x\in G} I(x)$, $\forall G \subset \mathcal{X}$ such that G is open.

- Idea: For a Borel set $E \subset \mathcal{X}$, we have $\mathbb{P}(X^N \in E) \sim \exp\left(-N \inf_{x\in E} I(x)\right)$ as $N \to \infty$.

Introduction to Large Deviations

Definition

Let $\{X^N\}_{N\in\mathbb{N}}$ be a sequence of random variables taking values in a complete separable metric space $\mathcal{X}$. We say that $\{X^N\}$ satisfies the *large deviation principle* with rate function I if there exists $I : \mathcal{X} \to [0, \infty]$ with compact level sets such that

1. $\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(X^N \in F) \leq -\inf_{x\in F} I(x)$, $\forall F \subset \mathcal{X}$ such that F is closed
2. $\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}(X^N \in G) \geq -\inf_{x\in G} I(x)$, $\forall G \subset \mathcal{X}$ such that G is open.

- Idea: For a Borel set $E \subset \mathcal{X}$, we have $\mathbb{P}(X^N \in E) \sim \exp\left(-N \inf_{x\in E} I(x)\right)$ as $N \to \infty$.
- So if $X^N \xrightarrow{d} X$, and $X \notin E$, $\inf_{x\in E} I(x) > 0$ characterizes the exponential rate of decay of the rare event $X^N \in E$.

Dawson-Gärtner LDP (1)

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N})dW_t^i, \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$$

Theorem (D. A. Dawson and J. Gärtner, 1987)

$\{\mu^N\}_{N\in\mathbb{N}}$ satisfies the LDP with rate function $I^{DG} : C([0,T];\mathcal{P}(\mathbb{R}^d)) \to [0,+\infty]$ given by

$$I^{DG}(\mu) = \frac{1}{2}\int_0^T \sup_{\phi\in C_c^\infty(\mathbb{R}^d):\mathrm{den}\neq 0} \frac{|\langle \dot{\mu}(t) - L_{\mu(t)}^*\mu(t), \phi\rangle|^2}{\langle \mu(t), \|\nabla^\top\phi\sigma\sigma^\top\nabla\phi\|^2\rangle} dt$$

$$L_\nu\phi(x) := b(x,\nu)\cdot\nabla\phi(x) + \frac{1}{2}\sigma\sigma^\top(x) : \nabla^2\phi(x), \quad \nu\in\mathcal{P}(\mathbb{R}^d)$$

if $\phi \mapsto \langle \mu, \phi\rangle$ is absolutely continuous in the sense of distributions and $\mu(0) = \lim_{N\to\infty}\mu_0^N$, and $I^{DG}(\mu) = +\infty$ otherwise.

Dawson-Gärtner LDP (2)

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N})dW_t^i, \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$$

$$I^{DG}(\mu) = \frac{1}{2}\int_0^T \sup_{\phi \in C_c^\infty(\mathbb{R}^d):\mathrm{den}\neq 0} \frac{|\langle \dot{\mu}(t) - L^*_{\mu(t)}\mu(t), \phi\rangle|^2}{\langle \mu(t), \|\nabla^\top \phi \sigma\sigma^\top \nabla\phi\|^2\rangle} dt$$

- Proof method: General theorems on projective limits for large deviation systems and LDPs on dual vector spaces to get the result for IID diffusion processes, then Girsanov's theorem to show closeness to an IID system

Dawson-Gärtner LDP (2)

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N})dW_t^i, \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$$

$$I^{DG}(\mu) = \frac{1}{2}\int_0^T \sup_{\phi \in C_c^\infty(\mathbb{R}^d):\text{den}\neq 0} \frac{|\langle \dot{\mu}(t) - L_{\mu(t)}^* \mu(t), \phi\rangle|^2}{\langle \mu(t), \|\nabla^\top \phi \sigma \sigma^\top \nabla \phi\|^2\rangle} dt$$

- Proof method: General theorems on projective limits for large deviation systems and LDPs on dual vector spaces to get the result for IID diffusion processes, then Girsanov's theorem to show closeness to an IID system
- Can be expressed in terms of derivatives of the free energy associated to the limiting McKean-Vlasov Equation viewed as a gradient flow on Wasserstein space in some settings (S. Adams et al., 2013). In gradient setting, connected to HJB Equations in (J. Feng and G. Kurtz, 2006)

Dawson-Gärtner LDP (2)

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N})dW_t^i, \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$$

$$I^{DG}(\mu) = \frac{1}{2}\int_0^T \sup_{\phi \in C_c^\infty(\mathbb{R}^d):\text{den}\neq 0} \frac{|\langle \dot{\mu}(t) - L^*_{\mu(t)}\mu(t), \phi\rangle|^2}{\langle \mu(t), \|\nabla^\top \phi \sigma\sigma^\top \nabla\phi\|^2\rangle} dt$$

- Proof method: General theorems on projective limits for large deviation systems and LDPs on dual vector spaces to get the result for IID diffusion processes, then Girsanov's theorem to show closeness to an IID system
- Can be expressed in terms of derivatives of the free energy associated to the limiting McKean-Vlasov Equation viewed as a gradient flow on Wasserstein space in some settings (S. Adams et al., 2013). In gradient setting, connected to HJB Equations in (J. Feng and G. Kurtz, 2006)
- Transition events for Dawson's system estimated in (J. Garnier

Buhiradja-Dupuis-Fischer LDP (1)

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N}, \mu_t^N)dW_t^i, \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$$

Theorem (A. Budhiraja et al., 2012)

$\{\mu^N\}_{N\in\mathbb{N}}$ satisfies the LDP with rate function $I : C([0,T];\mathcal{P}(\mathbb{R}^d)) \to [0,+\infty]$ given by

$$I(\mu) = \inf_{u\in\mathcal{U}:\mu(t)=\mathcal{L}(\hat{X}_t^u), t\in[0,T]} \frac{1}{2}\mathbb{E}\left[\int_0^T |u(t)|^2 dt\right]$$

$$d\hat{X}_t^u = [b(\hat{X}_t^u, \mathcal{L}(\hat{X}_t^u)) + \sigma(\hat{X}_t^u, \mathcal{L}(\hat{X}_t^u))u(t)]dt + \sigma(\hat{X}_t^u, \mathcal{L}(\hat{X}_t^u))dW_t, \quad \hat{X}_0^u \sim \nu$$

if $\mu(0) = \nu = \lim_{N\to\infty}\mu_0^N$ and $I(\mu) = +\infty$ otherwise. Here $\mathcal{U}$ is an appropriate class of stochastic controls $u : [0,T] \to \mathbb{R}^m$.

Buhiradja-Dupuis-Fischer LDP (2)

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N}, \mu_t^N)dW_t^i, \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}}$$

$$I(\mu) = \inf_{u \in \mathcal{U}: \mu(t) = \mathcal{L}(\hat{X}_t^u), t \in [0,T]} \frac{1}{2}\mathbb{E}\left[\int_0^T |u(t)|^2 dt\right]$$

$$d\hat{X}_t^u = [b(\hat{X}_t^u, \mathcal{L}(\hat{X}_t^u)) + \sigma(\hat{X}_t^u, \mathcal{L}(\hat{X}_t^u))u(t)]dt + \sigma(\hat{X}_t^u, \mathcal{L}(\hat{X}_t^u))dW_t, \quad \hat{X}_0^u \sim \nu$$

- Proof method: Weak convergence approach of (P. Dupuis and R. S. Ellis, 1997)

Buhiradja-Dupuis-Fischer LDP (2)

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N}, \mu_t^N)dW_t^i, \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$$

$$I(\mu) = \inf_{u\in\mathcal{U}:\mu(t)=\mathcal{L}(\hat{X}_t^u), t\in[0,T]} \frac{1}{2}\mathbb{E}\left[\int_0^T |u(t)|^2 dt\right]$$

$$d\hat{X}_t^u = [b(\hat{X}_t^u, \mathcal{L}(\hat{X}_t^u)) + \sigma(\hat{X}_t^u, \mathcal{L}(\hat{X}_t^u))u(t)]dt + \sigma(\hat{X}_t^u, \mathcal{L}(\hat{X}_t^u))dW_t, \quad \hat{X}_0^u \sim \nu$$

- Proof method: Weak convergence approach of (P. Dupuis and R. S. Ellis, 1997)
- More general assumptions than (D. A. Dawson and J. Gärtner, 1987)

Buhiradja-Dupuis-Fischer LDP (2)

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N}, \mu_t^N)dW_t^i, \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$$

$$I(\mu) = \inf_{u \in \mathcal{U}: \mu(t) = \mathcal{L}(\hat{X}_t^u), t \in [0,T]} \frac{1}{2}\mathbb{E}\left[\int_0^T |u(t)|^2 dt\right]$$

$$d\hat{X}_t^u = [b(\hat{X}_t^u, \mathcal{L}(\hat{X}_t^u)) + \sigma(\hat{X}_t^u, \mathcal{L}(\hat{X}_t^u))u(t)]dt + \sigma(\hat{X}_t^u, \mathcal{L}(\hat{X}_t^u))dW_t, \quad \hat{X}_0^u \sim \nu$$

- Proof method: Weak convergence approach of (P. Dupuis and R. S. Ellis, 1997)
- More general assumptions than (D. A. Dawson and J. Gärtner, 1987)
- Can be written in terms of relative entropy with respect to the limiting McKean-Vlasov Equation (M. Fischer, 2014)

Buhiradja-Dupuis-Fischer LDP (2)

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_t^N)dt + \sigma(X_t^{i,N}, \mu_t^N)dW_t^i, \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$$

$$I(\mu) = \inf_{u\in\mathcal{U}:\mu(t)=\mathcal{L}(\hat{X}_t^u), t\in[0,T]} \frac{1}{2}\mathbb{E}\left[\int_0^T |u(t)|^2 dt\right]$$

$$d\hat{X}_t^u = [b(\hat{X}_t^u, \mathcal{L}(\hat{X}_t^u)) + \sigma(\hat{X}_t^u, \mathcal{L}(\hat{X}_t^u))u(t)]dt + \sigma(\hat{X}_t^u, \mathcal{L}(\hat{X}_t^u))dW_t, \quad \hat{X}_0^u \sim \nu$$

- Proof method: Weak convergence approach of (P. Dupuis and R. S. Ellis, 1997)
- More general assumptions than (D. A. Dawson and J. Gärtner, 1987)
- Can be written in terms of relative entropy with respect to the limiting McKean-Vlasov Equation (M. Fischer, 2014)
- $I = I^{DG}$ proved even when σ depends on μ in (Z. B. and K. Spiliopoulos, 2023)

Lions Differentiation

Consider:

$$\mathcal{P}_2(\mathbb{R}^d) := \{\mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty\}.$$

Lions Derivative (R. Carmona and F. Delarue, 2018) Section 5.2

- We say $u : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is **Lions-differentiable** at $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if there exists $\tilde{u} : L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; \mathbb{R}^d) \to \mathbb{R}$ such that $\tilde{u}(X) = u(\mathcal{L}(X)), \forall X \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; \mathbb{R}^d)$ and $\tilde{u}$ is Fréchet differentiable at X_0 with $\mu_0 = \mathcal{L}(X_0)$.

Lions Differentiation

Consider:

$$\mathcal{P}_2(\mathbb{R}^d) := \{\mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty\}.$$

Lions Derivative (R. Carmona and F. Delarue, 2018) Section 5.2

- We say $u : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is **Lions-differentiable** at $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if there exists $\tilde{u} : L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; \mathbb{R}^d) \to \mathbb{R}$ such that $\tilde{u}(X) = u(\mathcal{L}(X)), \forall X \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; \mathbb{R}^d)$ and $\tilde{u}$ is Fréchet differentiable at X_0 with $\mu_0 = \mathcal{L}(X_0)$.
- There is a μ_0 a.s. unique deterministic measurable function $\xi : \mathbb{R}^d \to \mathbb{R}^d$ such that $D\tilde{u}(X_0) = \xi(X_0)$. We denote this equivalence class of $\xi \in L^2(\mathbb{R}^d, \mu_0; \mathbb{R}^d)$ by $\partial_\mu u(\mu_0)$ and call $\partial_\mu u(\mu_0)[\cdot] : \mathbb{R}^d \to \mathbb{R}^d$ the **Lions derivative** of u at μ_0.

Lions Differentiation

Consider:

$$\mathcal{P}_2(\mathbb{R}^d) := \{\mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty\}.$$

Lions Derivative (R. Carmona and F. Delarue, 2018) Section 5.2

- We say $u : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is **Lions-differentiable** at $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if there exists $\tilde{u} : L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; \mathbb{R}^d) \to \mathbb{R}$ such that $\tilde{u}(X) = u(\mathcal{L}(X)), \forall X \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; \mathbb{R}^d)$ and $\tilde{u}$ is Fréchet differentiable at X_0 with $\mu_0 = \mathcal{L}(X_0)$.
- There is a μ_0 a.s. unique deterministic measurable function $\xi : \mathbb{R}^d \to \mathbb{R}^d$ such that $D\tilde{u}(X_0) = \xi(X_0)$. We denote this equivalence class of $\xi \in L^2(\mathbb{R}^d, \mu_0; \mathbb{R}^d)$ by $\partial_\mu u(\mu_0)$ and call $\partial_\mu u(\mu_0)[\cdot] : \mathbb{R}^d \to \mathbb{R}^d$ the **Lions derivative** of u at μ_0.
- Note that this definition is independent of the choice of X_0, $\tilde{u}$, and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

Linear Functional Derivative

Definition: Linear Functional Derivative

Let $p : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$, $\mathcal{P}_2(\mathbb{R}) = \{\mu \in \mathcal{P}(\mathbb{R}) : \int |z|^2 \mu(dz) < \infty\}$. $(z, \mu) \ni \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \mapsto \frac{\delta}{\delta m} p(\mu)[z] \in \mathbb{R}$ is called the **Linear Functional Derivative** of p if for all $\nu_1, \nu_2 \in \mathcal{P}_2(\mathbb{R}^d)$:

$$p(\nu_2) - p(\nu_1) = \int_0^1 \int_{\mathbb{R}} \frac{\delta}{\delta m} p((1-r)\nu_1 + r\nu_2)[z](\nu_2(dz) - \nu_1(dz))$$

Example: $b(\mu) = \int \beta(z)\mu(dz)$, then $\frac{\delta}{\delta m} b(\mu)[z] = \beta(z)$.

Relationship: $\partial_\mu p(\mu) = \nabla_z \frac{\delta}{\delta m} p(\mu)[z]$.

Lions Differentiation: Empirical Projection

Proposition- 5.35 and 5.91 of (R. Carmona and F. Delarue, 2018)

For $g : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ twice continuously Lions differentiable with C^1 first Lions derivative, we can define the empirical projection of g, as $g^N : \mathbb{R}^{dN} \to \mathbb{R}$ given by

$$g^N(x_1, ..., x_N) := g\Big(\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}\Big) = g(\mu_x^N).$$

Lions Differentiation: Empirical Projection

Proposition- 5.35 and 5.91 of (R. Carmona and F. Delarue, 2018)

For $g : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ twice continuously Lions differentiable with C^1 first Lions derivative, we can define the empirical projection of g, as $g^N : \mathbb{R}^{dN} \to \mathbb{R}$ given by

$$g^N(x_1, ..., x_N) := g\Big(\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}\Big) = g(\mu_x^N).$$

Then g^N is twice differentiable on $(\mathbb{R}^d)^N$, and for each $x_1, .., x_N \in \mathbb{R}^d$, $(i, j) \in \{1, ..., N\}^2$:

$$\nabla_{x_i} g^N(x_1, ..., x_N) = \frac{1}{N}\partial_\mu g(\mu_x^N)[x_i]$$

and

$$\nabla_{x_i}\nabla_{x_j} g^N(x_1, ..., x_N) = \frac{1}{N}\partial_z\partial_\mu g(\mu_x^N)[x_i]\mathbb{1}_{i=j} + \frac{1}{N^2}\partial_\mu^2 g(\mu_x^N)[x_i, x_j].$$