COUNTEREXAMPLES TO L^p COLLAPSING ESTIMATES

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Abstract. We show that certain L^2 space-time estimates for generalized density matrices which have been used by several authors in recent years to study equations of BBGKY or Hartree-Fock type, do not have non-trivial L^pL^q generalizations.

1. Introduction and main results

In recent years, effective equations approximating the evolution of a large number of interacting Bosons or Fermions have been studied extensively. The best known example is the celebrated work of Erdös, Schlein and Yau [5], [6].

Since that work, a number of authors have studied the related Gross-Pitaevskii or BBGKY hierarchies, or the Hartree-Fock or Hartree-Fock-Bogoliubov equations, using harmonic analysis techniques and space-time L^2 estimates for a suitable trace density of solutions of the linear Schrödinger equation. We call such estimates "collapsing estimates", and list several instances, all in 3 space dimensions (thus, $x \in \mathbb{R}^3$, etc.).

If

$$G(t,x,y,z) = e^{\frac{it(\Delta_x+\Delta_y-\Delta_z)}{2}} G_0, \tag{1}$$

then

$$\|\nabla_x G(t,x,x,x)\|_{L^2(dtdx)} \lesssim \|\nabla_x \nabla_y \nabla_z G_0(x,y,z)\|_{L^2(dxdydz)}. \tag{2}$$

This estimate was used in the study of the Gross-Pitaevskii or BBGKY hierarchies. See [11] (where the estimate originates), as well as [1], [3], [4].

Another related example is: if

$$\Lambda(t,x,y) = e^{\frac{it(\Delta_x+\Delta_y)}{2}} \Lambda_0, \tag{3}$$

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then

$$\| |\nabla|_x^{1/2} \Lambda(t,x,x) \|_{L^2(dtdx)} \lesssim \| |\nabla|_x^{1/2} |\nabla|_y^{1/2} \Lambda_0(x,y) \|_{L^2(dxdy)}. \tag{4}$$

This estimate is useful for the Hartree-Fock-Bogoliubov equations, see [9], [10].

Finally, if

$$\Gamma(t,x,y) = e^{\frac{it(\Delta_x - \Delta_y)}{2}} \Gamma_0, \tag{5}$$

then

$$\| |\nabla_x|^{\frac{1}{2}} \langle \nabla_x \rangle^{2\epsilon} \Gamma(t,x,x) \|_{L^2(dtdx)} \lesssim_\epsilon \| \langle \nabla_x \rangle^{\frac{1}{2}+\epsilon} \langle \nabla_y \rangle^{\frac{1}{2}+\epsilon} \Gamma_0(x,y) \|_{L^2(dxdy)}. \tag{6}$$

Such estimates are relevant to both the Hartree-Fock-Bogoliubov equations mentioned above, and Hartree-Fock. See Theorem 3.3 in [2].

We also mention the approach of [7], [8] which applies to equation (5) and allows a wide range of $L^p(dt)L^q(dx)$ estimates on the left hand side, but the right hand side of the inequality is estimated in a Schatten norm.

It is natural to ask whether one can replace the $L^2(dt)L^2(dx)$ norm on the left hand side of estimates (2), (4) or (6) by an $L^p(dt)L^q(dx)$ norm, while keeping the right hand side in a Sobolev norm, which is useful for applications to PDEs. One can trivially make p or q bigger than 2 by putting more derivatives on the right hand side, so the interesting question is if one can make p or q less than 2.

The main result of this note is that this is impossible.

We prove the following closely related results.

Theorem 1.1. *Let Λ be given by* (3)*, with $x, y \in \mathbb{R}^n$. Assume*

$$\| |\nabla|_x^{\alpha} \Lambda(t,x,x) \|_{L^p(dt)L^q(dx)} \lesssim \| \Lambda_0(x,y) \|_{H^s(dxdy)} \tag{7}$$

for some $\alpha \geq 0, s \geq 0$. Then $p \geq 2$ and $q \geq 2$.

Theorem 1.2. *Let Γ be given by* (5)*, with $x, y \in \mathbb{R}^n$. Assume*

$$\| |\nabla|_x^{\alpha} \Gamma(t,x,x) \|_{L^p(dt)L^q(dx)} \lesssim \| \Gamma_0(x,y) \|_{H^s(dxdy)} \tag{8}$$

for some $\alpha \geq 0, s \geq 0$. Then $p \geq 2$ and $q \geq 2$.

Theorem 1.3. *Let G be given by* (1)*, with $x, y, z \in \mathbb{R}^n$. Assume*

$$\| |\nabla|_x^{\alpha} G(t,x,x,x) \|_{L^p(dt)L^q(dx)} \lesssim \| G_0(x,y,z) \|_{H^s(dxdydz)} \tag{9}$$

for some $\alpha \geq 0, s \geq 0$. Then $p \geq 2$ and $q \geq 2$.

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2. Proofs

2.1. **Proof of Theorem 1.1.**

2.1.1. *Necessity of* $p \geq 2$. Let R be a large number (which will approach ∞ at the end of the proof). Let C be a fixed large number (depending on n). Let

$$F_0(x,y) = e^{-\frac{|x|^2+|y|^2}{2CR}}$$

so that

$$e^{\frac{it(\Delta_x+\Delta_y)}{2}} F_0 := F(t,x,y) = \frac{1}{(1+it/(CR))^n} e^{-\frac{|x|^2+|y|^2}{2(CR+it)}}. \tag{10}$$

We think of $F(t,x,y)$ as the basic "vertical tube" solution to the linear Schrödinger equation in $2n+1$ dimensions which is essentially 1 if $|x|,|y| \leq R^{1/2}$, $0 \leq t \leq R$. The rigorous statement is that C is chosen so that $\Re F(t,x,y) \geq \frac{1}{2}$ in the above range. Also, the Fourier transform (in space) of F is essentially supported at frequencies $|\xi|,|\eta| \leq R^{-1/2}$.

We choose the function $\Lambda(t,x,y)$ to be a sum of translates and modulations of $F(t,x,y)$ which are inclined at 45 degrees and are trained to reach the region $|x| \leq \frac{1}{100}$, $|y| \leq \frac{1}{100}$, $R - R^{\frac{1}{2}} < t < R$ with almost the same oscillation (and almost no cancellations). The summands will have Fourier transforms essentially supported in balls of radius $R^{-1/2}$ centered at unit vectors.

Explicitly, choose roughly $R^{n-\frac{1}{2}}$ points (x_k,y_k) which are spaced at distance $R^{1/2}$ from each other on the sphere $|(x,y)| = R$. For technical reasons, we only choose points for which all coordinates are $\geq \frac{R}{10n}$. Define $(\xi_k,\eta_k) = \frac{(x_k,y_k)}{R}$.

Choose the following initial conditions:

$$\Lambda_0(x,y) = \sum e^{i(x\cdot\xi_k+y\cdot\eta_k)} F_0(x+x_k, y+y_k).$$

The functions being summed are approximately orthogonal and each have L^2 norm $\sim R^{n/2}$:

$$\int \left|F_0(x+x_k,y+y_k)F_0(x+x_l,y+y_l)\right| dxdy = \pi^n (CR)^n e^{-\frac{|(x_k,y_k)-(x_l,y_l)|^2}{4CR}}. \tag{11}$$

Recalling that the sum has $\sim R^{n-\frac{1}{2}}$ terms, we derive

$$\|\Lambda_0\|_{L^2(dxdy)} \lesssim R^{n-\frac{1}{4}}.$$

The same type of upper bound holds for higher order derivatives (since $|(\xi_k,\eta_k)| = 1$), thus, for each fixed s,

$$\|\Lambda_0\|_{H^s(dxdy)} \lesssim R^{n-\frac{1}{4}}. \tag{12}$$

The solution looks like

$$\Lambda(t,x,y) = \sum e^{-it\frac{(|\xi_k|^2+|\eta_k|^2)}{2}} e^{i(x\cdot\xi_k+y\cdot\eta_k)} F(t, x+x_k - t\xi_k, y+y_k-t\eta_k)$$
$$= e^{-i\frac{t}{2}} \sum e^{i(x\cdot\xi_k+y\cdot\eta_k)} F(t, x+x_k - t\xi_k, y+y_k-t\eta_k),$$

and

$$\big|\Lambda(t,x,y)\big| \geq \Re \sum e^{i(x\cdot\xi_k+y\cdot\eta_k)} F(t, x+x_k - t\xi_k, y+y_k-t\eta_k) \sim R^{n-\frac{1}{2}},$$

if $|(x,y)| \leq \frac{1}{100}$, $R - R^{\frac{1}{2}} < t < R$. Thus

$$R^{\frac{1}{2p}} R^{n-\frac{1}{2}} \lesssim \|\Lambda(t,x,x)\|_{L^p(dt)L^q(dx)}, \tag{13}$$

so, recalling (12), if

$$\|\Lambda(t,x,x)\|_{L^p(dt)L^q(dx)} \lesssim \|\Lambda_0(x,y)\|_{H^s(dxdy)},$$

then $p \geq 2$.

Using the product rule and the lower bounds on the components of ξ_k, η_k, same argument works for ordinary derivatives of order $\alpha = m \in \mathbb{N}$.

To justify the statement for fractional derivatives of non-integer order α, do a Littlewood-Paley decomposition in space $\Lambda(t,\cdot,\cdot) = P_{\leq 10}\Lambda(t,\cdot,\cdot) + P_{\geq 10}\Lambda(t,\cdot,\cdot)$, where $P_{\leq 10}$ localizes functions of $2n$ variables, smoothly at frequencies ≤ 10. Then $P_{\geq 10}\Lambda(t,\cdot,\cdot)$ is exponentially small as $R \to \infty$. This is true for the function F_0, and its translates by a unit vector in Fourier space.

A crude estimate is

$$\|P_{\geq 10}\Lambda(t,\cdot,\cdot)\|_{H^s} \lesssim_s e^{-\sqrt{R}}.$$

For our counterexample, we use $P_{\leq 10}\Lambda(t,\cdot,\cdot)$ instead of $\Lambda(t,\cdot,\cdot)$.

Thus, for R sufficiently large, $|\nabla^m P_{\leq 10}\Lambda(t,x,y)| \sim |\nabla^m \Lambda(t,x,y)| \sim R^{n-\frac{1}{2}}$ if $|(x,y)| \leq \frac{1}{100}$, $R - R^{\frac{1}{2}} < t < R$. The function $(P_{\leq 10}\Lambda)(t,x,x)$ is supported, in Fourier space, at frequencies $|\xi| \leq 20$. Denote, by abuse of notation, $P_{\leq 20}$ the operator localizing functions of n variables at frequencies $|\xi| \leq 20$. Let $m \in \mathbb{N}$, $m > \alpha$. Then the operator $\frac{\nabla^m}{|\nabla|^\alpha} P_{\leq 20}$ (defined in the obvious way on the Fourier transform side) is bounded on all L^p spaces, and

$$R^{\frac{1}{2p}} R^{n-\frac{1}{2}} \lesssim \|\nabla^m (P_{\leq 10}\Lambda)(t,x,x)\|_{L^p(dt)L^q(dx)}$$
$$= \|\frac{\nabla^m}{|\nabla|^\alpha} P_{\leq 20} |\nabla|^\alpha (P_{\leq 10}\Lambda)(t,x,x)\|_{L^p(dt)L^q(dx)}$$
$$\lesssim \||\nabla|^\alpha (P_{\leq 10}\Lambda)(t,x,x)\|_{L^p(dt)L^q(dx)},$$

while

$$\|P_{\leq 10}\Lambda_0\|_{H^s(dxdy)} \lesssim C^n R^{n-\frac{1}{4}}.$$

Letting $R \to \infty$, we conclude $p \geq 2$ as before.

2.1.2. *Necessity of* $q \geq 2$. Let $F(t,x,y)$ be the basic vertical tube solution of height R (as (10)). Let $m \gg 1$. Choose roughly $R^{mn-\frac{n}{2}}$ points x_k which are spaced at distance $\sim R^{\frac{1}{2}}$ in a large ball $B(0,R^m)$ of radius R^m in $\mathbb{R}^n$. Fix a unit vector $\xi \in S^{n-1}$.

We take initial conditions

$$\Lambda_0(x,y) = e^{i(x+y)\cdot\xi} \sum F_0(x+x_k, y+x_k).$$

Then

$$\Lambda(t,x,y) = e^{i(x+y)\cdot\xi} e^{-it} \sum F(t, x+x_k-t\xi, y+x_k-t\xi).$$

There are roughly $R^{mn-\frac{n}{2}}$ terms in the sum. The summands are essentially orthogonal (as in (11)) and each term has L^2 norm $\sim R^{n/2}$, thus

$$\|\Lambda_0\|_{L^2(dxdy)} \sim R^{\frac{n}{4}+\frac{mn}{2}}.$$

On the other hand, each $F(t, x+x_k-t\xi, y+x_k-t\xi)$ is essentially 1 on a tube T_k of radius $R^{1/2}$ and length R in $2n+1$ dimensions, and rapidly decaying out of T_k. Note that at $t=0$, T_k is centered at $(0,-x_k,-x_k)$. Moreover, these tubes T_k are in the same direction $(1,\xi,\xi)$ and hence disjoint. Therefore, $|\Lambda(t,x,y)| \gtrsim 1$ on the union of the tubes T_k. In particular, $|\Lambda(t,x,x)| \gtrsim 1$ for $0 \leq t \leq R$ and $x \in B(t\xi, R^m)$. We only need the previous estimate for $0 \leq t \leq 1$, where the claim is obvious. In addition, the Fourier transform of $\Lambda(t,x,x)$ is supported (essentially) in a $R^{-\frac{1}{2}}$ neighbourhood of the point 2ξ , with $|\xi| = 1$, so $||\nabla|^\alpha \Lambda(t,x,x)| \gtrsim 1$ for $0 \leq t \leq 1$ and $x \in B(t\xi, R^m)$. Thus

$$\||\nabla|^\alpha \Lambda(t,x,x)\|_{L^p([0,1])L^q(dx)} \gtrsim R^{\frac{mn}{q}},$$

while $\|\Lambda_0\|_{H^s(dxdy)} \sim \|\Lambda_0\|_{L^2(dxdy)} \sim R^{\frac{n}{4}+\frac{mn}{2}}$ and $m \gg 1$, so $q \geq 2$ is necessary.

2.2. **Proof of Theorem 1.2.** The examples for Γ are similar to those for Λ, and are included for completeness.

2.2.1. *Necessity of* $p \geq 2$. First we take the basic "vertical tube" solution. Let

$$F_0(x,y) = e^{-\frac{|x|^2+|y|^2}{2CR}}$$

so that

$$e^{\frac{it(\Delta_x-\Delta_y)}{2}} F_0 := F(t,x,y) = \frac{1}{(1+(\frac{t}{CR})^2)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2(CR+it)}} e^{-\frac{|y|^2}{2(CR-it)}}. \qquad (14)$$

The solution $F(t,x,y)$ is essentially 1 if $|x|, |y| \leq R^{1/2}$, $0 \leq t \leq R$. More precisely, we choose a large constant $C = C(n)$ so that $\Re F(t,x,y) \geq \frac{1}{2}$ in the above range. Also, as before, the Fourier transform (in space) of F is essentially supported at frequencies $|\xi|, |\eta| \leq R^{-1/2}$.

Pick roughly $R^{n-\frac{1}{2}}$ points (x_k, y_k) which are spaced at distance $\sim R^{1/2}$ from each other on the surface $\{(x,y) : |x| = |y|, \frac{R}{2} \leq |x| \leq R\}$. Define $(\xi_k, \eta_k) = \frac{1}{R}(x_k, y_k)$ so that $|\xi_k|^2 - |\eta_k|^2 = 0$ and $|(\xi_k, \eta_k)| \sim 1$.

Take the following initial conditions

$$\Gamma_0(x,y) = \sum e^{i(x\cdot\xi_k - y\cdot\eta_k)} F_0(x + x_k, y + y_k)$$

so that the solution is

$$\begin{aligned}\Gamma(t,x,y) &= \sum e^{-it\frac{(|\xi_k|^2-|\eta_k|^2)}{2}} e^{i(x\cdot\xi_k - y\cdot\eta_k)} F(t, x + x_k - t\xi_k, y + y_k - t\eta_k) \\ &= \sum e^{i(x\cdot\xi_k - y\cdot\eta_k)} F(t, x + x_k - t\xi_k, y + y_k - t\eta_k).\end{aligned}$$

Since the $\sim R^{n-\frac{1}{2}}$ terms in Γ_0 are essentially orthogonal and each have L^2 norm $\sim R^{n/2}$, we get

$$\|\Gamma_0\|_{L^2(dxdy)} \lesssim R^{n-\frac{1}{4}}.$$

Moreover, since $|(\xi_k, \eta_k)| \sim 1$, there also holds

$$\|\Gamma_0\|_{H^s(dxdy)} \lesssim R^{n-\frac{1}{4}}. \qquad (15)$$

From the expression of Γ, we see that

$$|\Gamma(t,x,y)| \gtrsim R^{n-\frac{1}{2}} \quad \text{for } |(x,y)| \leq \frac{1}{100}, R - R^{\frac{1}{2}} < t < R.$$

Therefore,

$$\|\Gamma(t,x,x)\|_{L^p(dt)L^q(dx)} \gtrsim R^{\frac{1}{2p}} R^{n-\frac{1}{2}},$$

so, recalling (15), if

$$\|\Gamma(t,x,x)\|_{L^p(dt)L^q(dx)} \lesssim \|\Gamma_0(x,y)\|_{H^s(dxdy)},$$

then $p \geq 2$. From a similar argument to the one in subsection 2.1.1 (i.e. only using x_k, y_k for which all coordinates of ξ_k and $-\eta_k$ are $\geq \frac{1}{10n}$), $p \geq 2$ is also necessary for estimates of the form

$$\| |\nabla|_x^\alpha \Gamma(t,x,x) \|_{L^p(dt)L^q(dx)} \lesssim \|\Gamma_0(x,y)\|_{H^s(dxdy)}.$$

2.2.2. *Necessity of* $q \geq 2$. Let $F(t,x,y)$ be the basic vertical tube solution of height R (as (14)). Let $m \gg 1$. Choose roughly $R^{mn-\frac{n}{2}}$ points x_k which are spaced at distance $\sim R^{\frac{1}{2}}$ in a large ball $B(0,R^m)$ of radius R^m in $\mathbb{R}^n$. Fix a unit vector $\xi \in S^{n-1}$.

We take initial conditions

$$\Gamma_0(x,y) = e^{ix\cdot\xi} \sum F_0(x+x_k, y+x_k),$$

so that the solution is

$$\Gamma(t,x,y) = e^{ix\cdot\xi} \sum F(t, x+x_k - t\xi, y+x_k).$$

Note that $\Gamma(t,x,x) \gtrsim 1$ for $0 \leq t \leq 1$ and $|x| \leq R^m$. Moreover, the Fourier transform of $\Gamma(t,x,x)$ is essentially supported in a $R^{-1/2}$ neighborhood of the point ξ with $|\xi| = 1$.

Then, the necessity of $q \geq 2$ follows from the same calculation as in subsection 2.1.2.

2.3. **Proof of Theorem 1.3.** The examples for G are similar to those in previous subsections.

2.3.1. *Necessity of* $p \geq 2$. First we take the basic "vertical tube" solution. Let

$$F_0(x,y,z) = e^{-\frac{|x|^2+|y|^2+|z|^2}{2CR}}$$

so that

$$\begin{aligned} e^{\frac{it(\Delta_x+\Delta_y-\Delta_z)}{2}} F_0 &:= F(t,x,y,z) \\ &= \frac{1}{(1+\frac{it}{CR})^n (1-\frac{it}{CR})^{\frac{n}{2}}} e^{-\frac{|x|^2+|y|^2}{2(CR+it)}} e^{-\frac{|z|^2}{2(CR-it)}}. \end{aligned} \tag{16}$$

The solution $F(t,x,y,z)$ is essentially 1 if $|(x,y,z)| \leq R^{1/2}$, $0 \leq t \leq R$. Also, the Fourier transform (in space) of F is essentially supported at frequencies $|(\xi,\eta,\zeta)| \leq R^{-1/2}$.

Pick roughly $R^{\frac{3n-1}{2}}$ points (x_k,y_k,z_k) which are spaced at distance $\sim R^{1/2}$ from each other on the surface $\{(x,y,z) : |x|^2+|y|^2 = |z|^2, \frac{R}{2} \leq |x|, |y| \leq R\}$. Define $(\xi_k,\eta_k,\zeta_k) = \frac{1}{R}(x_k,y_k,z_k)$ so that

$$|\xi_k|^2 + |\eta_k|^2 = |\zeta_k|^2 \quad \text{and} \quad |(\xi_k,\eta_k,\zeta_k)| \sim 1.$$

Take the following initial conditions

$$G_0(x,y,z) = \sum e^{i(x\cdot\xi_k + y\cdot\eta_k - z\cdot\zeta_k)} F_0(x+x_k, y+y_k, z+z_k)$$

so that the solution is

$$\begin{aligned} &G(t,x,y,z) \\ &= \sum e^{i(x\cdot\xi_k + y\cdot\eta_k - z\cdot\zeta_k)} F(t, x+x_k - t\xi_k, y+y_k - t\eta_k, z+z_k - t\zeta_k), \end{aligned}$$

since $|\xi_k|^2 + |\eta_k|^2 = |\zeta_k|^2$.

Since the $\sim R^{\frac{3n-1}{2}}$ terms in G_0 are essentially orthogonal and each has L^2 norm $\sim R^{3n/4}$, we get

$$\|G_0\|_{L^2(dxdydz)} \lesssim R^{\frac{3n}{2}-\frac{1}{4}}.$$

Moreover, since $|(\xi_k, \eta_k, \zeta_k)| \sim 1$, there also holds

$$\|G_0\|_{H^s(dxdydz)} \lesssim R^{\frac{3n}{2}-\frac{1}{4}}. \tag{17}$$

From the expression of G, we see that

$$|G(t,x,y,z)| \gtrsim R^{\frac{3n-1}{2}} \quad \text{for } |(x,y,z)| \leq \frac{1}{100}, R - R^{\frac{1}{2}} < t < R.$$

Therefore,

$$\|G(t,x,x,x)\|_{L^p(dt)L^q(dx)} \gtrsim R^{\frac{1}{2p}} R^{\frac{3n-1}{2}}.$$

Recalling (17), if

$$\|G(t,x,x,x)\|_{L^p(dt)L^q(dx)} \lesssim \|G_0(x,y,z)\|_{H^s(dxdydz)},$$

then $p \geq 2$. From a similar argument as in subsection 2.1.1, $p \geq 2$ is also necessary for estimates of the form

$$\||\nabla|_x^\alpha G(t,x,x,x)\|_{L^p(dt)L^q(dx)} \lesssim \|G_0(x,y,z)\|_{H^s(dxdydz)}.$$

2.3.2. *Necessity of $q \geq 2$.* Let $F(t,x,y,z)$ be the basic vertical tube solution of height R (as (16)). Let $m \gg 1$. Choose roughly $R^{mn-\frac{n}{2}}$ points x_k which are spaced at distance $\sim R^{\frac{1}{2}}$ in a large ball $B(0,R^m)$ of radius R^m in $\mathbb{R}^n$. Fix a unit vector $\xi \in S^{n-1}$.

We take initial conditions

$$G_0(x,y,z) = e^{i(x+y-z)\cdot\xi} \sum F_0(x+x_k, y+x_k, z+x_k),$$

so that the solution is

$$\begin{aligned} &G(t,x,y) \\ &= e^{\frac{-it}{2}} e^{i(x+y-z)\cdot\xi} \sum F(t, x+x_k - t\xi, y+x_k - t\xi, z+x_k - t\xi). \end{aligned}$$

There are roughly $R^{mn-\frac{n}{2}}$ terms in the sum. The summands are essentially orthogonal and each term has L^2 norm $\sim R^{3n/4}$, thus

$$\|G_0\|_{L^2(dxdydz)} \sim R^{\frac{n}{2}+\frac{mn}{2}}.$$

On the other hand, each $F(t, x + x_k - t\xi, y + x_k - t\xi, z + x_k - t\xi)$ is essentially 1 on a tube T_k of radius $R^{1/2}$ and length R in $3n + 1$ dimensions, and rapidly decaying out of T_k. Note that at $t = 0$, T_k is centered at $(0, -x_k, -x_k, -x_k)$. Moreover, these tubes T_k are in the same direction $(1, \xi, \xi, \xi)$ and hence disjoint. Therefore, $|G(t, x, y, z)| \gtrsim 1$ on the union of the tubes T_k. In particular, $|G(t, x, x, x)| \gtrsim 1$ for $0 \le t \le R$ and $x \in B(t\xi, R^m)$. Thus

$$\|G(t, x, x, x)\|_{L^p([0,1])L^q(dx)} \gtrsim R^{\frac{mn}{q}}$$

(with a similar estimate for $|\nabla|^\alpha G(t, x, x, x)$), while $\|G_0\|_{H^s(dxdy)} \sim R^{\frac{n}{2}+\frac{mn}{2}}$ and $m \gg 1$, so $q \ge 2$ is necessary.

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