Math 130 – Spring 2015 – Boyle –Exam 3

- NO CALCULATORS OR ELECTRONIC DEVICES ALLOWED.
- Use a separate answer sheet for each question.
- Give your pledge on page 1 only, covering the whole test.
- Draw a box around a final answer to a problem.

1. (14 points)

Let f be the function $f(x) = x^2 - 8\ln x$ with domain $[1, 10]$.

(a) (4 pts) What properties of f and its domain guarantee that f will assume maximum and minimum values?

Solution.

f is continuous and the domain is a finite closed interval.

(b) (10 pts) What are the maximum and minimum values assumed by f on its domain?

Solution.

$f'(x) = 2x - (8/x)$. So, $f'(x) = 0$ at $x = 2$.

Because f is differentiable, the max and min values can only be assumed at inputs from $\{1, 2, 10\}$. So,

Minimum value is $f(2) = 4 - 8\ln(2)$
(by the first derivative test, or by comparing values).

Maximum value is $f(10) = 100 - 8\ln(10)$
(this number is larger than $f(1) = 1$).

2. (10 points)

Find the equation of the tangent line to the curve $4e^{2x} - y^2 = 0$ at the point $(0, 2)$.

Solution.

Use implicit differentation. Differentiating with respect to x:

$$\begin{aligned} 4e^{2x} - y^2 &= 0 \\ 8e^{2x} - 2yy' &= 0 \\ 2yy' &= 8e^{2x} \\ y' &= \frac{4e^{2x}}{y} \ . \end{aligned}$$

For $(x, y) = (0, 2)$, we have $y' = 4e^0/2 = 2$, and an equation for that tangent line is

$$y - 2 = 2x \ .$$

3. (15 points)
Let f be the function with domain $[0,2]$ defined by $f(x) = \sqrt{2x+1}$.

(a) (7 pts) Compute the left endpoint Riemann sum estimate $\sum_{i=1}^{4} f(x_{i-1})\Delta x$ of $\int_{x=0}^{2} f(x)\,dx$ when $n = 4$. (Do not simplify the expression you obtain from the definition.)
Solution.

$$\sqrt{2(0)+1}\,(1/2)+\sqrt{2(1/2)+1}\,(1/2)+\sqrt{2(1)+1}\,(1/2)+\sqrt{2(3/2)+1}\,(1/2)\ .$$

(b) (5 pts) Draw the graph of f and the rectangles corresponding to this Riemann sum.
Solution.
Not included for technical reasons.

(c) (3 pts) Is this Riemann sum greater or smaller than $\int_{x=0}^{2} f(x)\,dx$?
Solution.
Smaller.

4. (14 points)
Let f be the function on $[0,4]$ defined by $f(x) = (2x+1)^{1/4}$. Let R be the "region under the curve", i.e. the set of points (x,y) such that $0 \leq x \leq 4$ and $0 \leq y \leq f(x)$. Let S be the solid of revolution obtained by rotating R about the x-axis.

What is the volume of S?

Solution.

$$\begin{aligned}
\text{volume}(S) &= \int_{x=0}^{4} \pi[f(x)]^2 \\
&= \int_{x=0}^{4} \pi(2x+1)^{1/2} \\
&= \pi\Big[(1/3)(2x+1)^{3/2}\Big]_{x=0}^{4} \\
&= \pi\Big((1/3)(9^{3/2} - (1/3)(1))\Big) \\
&= \frac{\pi}{3}(27-1) \\
&= \frac{26\pi}{3} \ .
\end{aligned}$$

5. (18 points)

(a) (8 pts) Compute the average value of the function $f(x) = \sec^2(x)$ over the interval $[0, \pi/4]$.

Solution.

This average value $\overline{f}$ is

$$\begin{aligned}\overline{f} &= \frac{1}{(\pi/4)} \int_{x=0}^{\pi/4} \sec^2(x)\,dx \\ &= \frac{4}{\pi}\Big[\tan(x)\Big]_{x=0}^{\pi/4} \\ &= \frac{4}{\pi}\big(\tan(\pi/4) - \tan(0)\big) \\ &== \frac{4}{\pi}(1-0) = \frac{4}{\pi}\ .\end{aligned}$$

(b) (10 pts) Evaluate the definite integral

$$\int_{x=\pi/4}^{\pi/2} \sqrt{\sin x}\,\cos x\,dx$$

Solution.

We use a substitution $u(x) = u = \sin(x)$. Then $du/dx = \cos x$, and

$$\begin{aligned}\int_{x=\pi/4}^{\pi/2} \sqrt{\sin x}\,\cos x\,dx \;&=\; \int_{u=u(\pi/4)}^{u(\pi/2)} \sqrt{u}\,du \\ = \int_{u=1/\sqrt{2}}^{1} \sqrt{u}\,du \;&=\; \left[\frac{2}{3}u^{3/2}\right]_{u=1/\sqrt{2}}^{1} \\ = \frac{2}{3} - \frac{2}{3}(1/\sqrt{2})^{3/2} \\ = \frac{2}{3}(1 - 2^{-3/4})\ .\end{aligned}$$

6. (14 points)
Let $s(t)$ be the position of a certain object at time t. Suppose its velocity at time t is e^{2t}, and suppose $s(0) = 1$.
What is the position of the object at time $t = 3$?
Solution.

$$\begin{aligned} s(3) - s(0) = \int_{t=0}^{3} e^{2t}\, dt \;&=\; \Big[(1/2)e^{2t}\Big]_{t=0}^{3} \\ =&(1/2)e^{(2)3} - (1/2)e^{(2)0} \\ =&(e^6 - 1)/2 \;. \end{aligned}$$

Therefore

$$\begin{aligned} s(3) &= s(0) + (e^6 - 1)/2 \\ &= 1 + (e^6 - 1)/2 \\ &= \frac{e^6 + 1}{2} \;. \end{aligned}$$

7. (15 points) According to Poiseuille's laws, the velocity v of blood in a blood vessel is given by $v(r) = k(R^2 - r^2)$, where R is the (constant) radius of the blood vessel, r is the distance of the flowing blood from the center of the blood vessel, and k is a positive constant.

Given R, let $Q(R)$ be the total blood flow (in milliliter per minute) in the vessel. For n a positive integer, $Q(R)$ is approximated by a sum

$$\sum_{i=1}^{n} v(r_i) 2\pi r_i \Delta r$$

in which $\Delta R = R/n$ and $r_i = i\Delta r$. As n goes to ∞, the sum converges to $Q(R)$.

(a) (5 pts) Write a definite integral which equals $Q(R)$.

Solution.

$$\int_{r=0}^{R} v(r) 2\pi r \, dr \ , \text{ which equals } \int_{r=0}^{R} k(R^2 - r^2) 2\pi r \, dr \ .$$

(b) (10 pts) Compute the definite integral.

Solution.

$$\begin{aligned}
\int_{r=0}^{R} k(R^2 - r^2) 2\pi r \, dr &= 2\pi k \int_{r=0}^{R} (R^2 - r^2) r \, dr \\
&= 2\pi k \int_{r=0}^{R} R^2 r - r^3 \, dr \\
&= 2\pi k \Big[(1/2) R^2 r^2 - (1/4) r^4 \Big]_{r=0}^{R} \\
&= 2\pi k \big((1/2) R^4 - (1/4) R^4 \big) \\
&= 2\pi k (1/4) R^4 = \ \pi k (1/2) R^4 \\
&= \frac{\pi k R^4}{2} \ .
\end{aligned}$$