Matrices for solving linear systems of equations

Below is a quick overview of what we did in lecture on Section 10.1. Not all of this material is in the book (and not everything in the book was in the lecture). What's below doesn't contain the lecture – if you weren't there, please see a classmate's notes, and talk with a classmate or TA or me if you have a question.

On to the overview. What you should know and could be tested on.

The definition of a linear system of equations. The definition of the solution set of such a system. The three possibilities for a solution set of a linear system: an infinite set; a set containing exactly one solution; the empty set.

The strategy for systematically solving a linear system: apply elementary moves which replace the system with a more simple system which has the same solution set. This can be done by applying elementary row operations (there are three types) to the augmented matrix of a linear system – this will produce the augmented matrix of the corresponding simplified linear system. Keep doing this until arriving at a system with an "obvious" type of solution set. (Section 10.1 describes this in detail.)

Now in class we did a little more from here about what kind of matrix made the type of solution "obvious", and easy to solve for. This kind of matrix is one with the "staircase" (echelon) form. Say a lead entry is a nonzero entry which is the leftmost nonzero entry in its row. That the matrix is in the "staircase" (echelon) form means that for any $i > 1$, if row i has lead entry in column j, then row $(i-1)$ has a lead entry in a column to the left of column j. (In particular, if some row is a zero row (every entry zero), then all lower rows must also be zero rows.)

For an $m \times (n+1)$ augmented matrix M, the corresponding linear system consists of m linear equations in n unknowns (which I'll call $x_1, \ldots, x_n$). If M is in staircase form, then

- The corresponding linear system has a solution if and only if there is no lead term in the last column (column $n+1$).

Now, suppose the system does have a solution. Say x_k is a *free variable* if column k does not contain a lead entry; x_k is a *basic variable* if column k does contain a lead entry. Then

- For each assignment of numerical values to the free variables, there is a unique assignment of numerical values to the basic variables which gives a solution to the system.

- So, given that a solution exists, there are infinitely many solutions if and only if there is at least one free variable.

When an augmented matrix is in echelon form, we can do more elementary row operations to put it into an even nicer form: *reduced* echelon form. A matrix is in reduced echelon form if (i) it is in echelon form, (ii) every lead entry is 1 and (iii) each lead entry is the only nonzero entry in its column.

For example, below is an augmented matrix in reduced echelon form:

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & -.77 \end{pmatrix}$$

The corresponding linear system (in variables $x_1, x_2, \ldots, x_6$) is

$$\begin{aligned} x_2 + 2x_3 - 2x_6 &= 2 \\ x_4 &= 3 \\ x_5 + x_6 &= -.77 \end{aligned}$$

Here the free variables are x_1, x_3, x_6. It is particularly easy from this form to write the basic variables a functions of free variables:

$$\begin{aligned} x_2 &= 2 - 2x_3 + 2x_6 \\ x_4 &= 3 \\ x_5 &= -.77 - x_6 \end{aligned}$$

Now it's easy to see that for any assignment of numerical values to the free variables, there is a unique assignment of numerical values to the basic variables which gives a solution to the system.

What might I ask you about from this on an exam? For example ...

For computation, I could ask you to take a linear system, write down its augmented matrix, apply elementary row and column operations to put it into the staircase (echelon) form, and give a solution, or perhaps just tell me how many solutions there are.

Or, I could give you matrices in echelon form which are augmented matrices, and ask you to tell me how many solutions there are, or which variables are free variables.

Or, I could ask you to write down the linear system corresponding to a given augmented matrix.

Or, ask you something about the material above in some way (e.g., what is the definition of a solution set to a linear system? does every linear system have a solution set? what are the three types of elementary row operation?)

We of course can only solve small linear systems by hand (and I will not give an example problem which requires you to do a lot of arithmetic). But if you understand this material, you'll have an idea of what computers do to systematically solve systems of many equations in many unknowns, and what the solutions mean.

Sample problem.

For the matrices named below ...

1. Circle the entries which are lead terms.
2. Which matrices are in the echelon form?
3. Consider them as augmented matrices. How many solutions are there to the corresonding linear systems? If the variables are $x_1, x_2, \ldots$, which are the free variables?

$$A = \begin{pmatrix} 0 & 8 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 8 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$