Math 410 – Spring 2016 – Boyle –Exam 1 Solutions

There are 100 points possible.

1(a). (5 points) State the Extreme Value Theorem.
Solution. See the text for the statement.

1(b). (5 points) State the Mean Value Theorem.
Solution. See the text for the statement.

1(c). (6 points) State the positivity axioms for the real numbers.
Solution.
There is a subset P of $\mathbb{R}$ (the positive set) such that the following hold.
(i) For all a and b in P, $a + b \in P$ and $ab \in P$.
(ii) (Trichotomy) For every a in $\mathbb{R}$, exactly one of the following holds: $a \in P$; $-a \in P$; $a = 0$.
(Note, if you drop "exactly", then the meaning changes drastically: properties (i) and (ii) would hold if P were defined to be $\mathbb{R}$.)

2(a). (5 points) Suppose D is a subset of $\mathbb{R}$, $f\colon D \to \mathbb{R}$ and $x_0 \in D$. State the epsilon - delta condition for f to be continuous at x_0.
Solution.
For every $\epsilon > 0$ there exists $\delta > 0$ such that for every x in D,

$$|x - x_0| < \delta \implies |f(x) - f(x_0) < \epsilon \ .$$

2(b). (5 points) Now define $D = [0, 2]$ with $f(x) = 5$ if $x \leq 1$ and $f(x) = 7$ if $x > 1$. Prove that f does not satisfy the epsilon-delta condition for continuity at the point $x_0 = 1$.
Solution.
Set $\epsilon = 1$. Given $\delta > 0$, set $x = 1 + \delta/2$. Then $|x - 1| = \delta/2 < \delta$, but $|f(x) - f(1)| = |7 - 5| > \epsilon$.

3(a). (5 points) Compute $\liminf\ (-1)^n(2 - e^{-n})^3$. No proof required.
Solution.
-8 .

3(b). (5 points) Give an example of bounded sequences of real numbers, (a_n) and (b_n), for which $\limsup(a_n + b_n) \neq \limsup(a_n) + \limsup(b_n)$. State what the numbers $\limsup(a_n)$ and $\limsup(b_n)$ are in your example. No proof required.

Solution.

Set $a_n = (-1)^n$, $b_n = (-1)^{n+1}$. Then $\limsup(a_n) = 1$, $\limsup(b_n) = 1$ and $\limsup(a_n + b_n) = 0$.

4. (7 points) Let (C) denote the following condition on a sequence (b_n) of real numbers and a real number b: there exists $\epsilon > 0$ such that there exists N such that for all indices $n > N$, $|b_n - b| < \epsilon$.

Which of the rules (i),(ii),(iii),(iv) below define sequences $(b_n)_{n=1}^{\infty}$ which satisfy condition (C) for some real number b? No proof required.

(i) $b_n = 4 + (8/n)$

(ii) $b_n = (-1)^n(4 + (8/n))$

(iii) $b_n = 1$ if $n \leq 23$ and $b_n = 2$ if $n > 23$

(iv) $b_n = \sqrt{n}$

Solution.

A sequence (b_n) satisfies condition (C) if and only it is bounded, i.e., there is a real number M such that for all n, $|b_n| \leq M$. This holds for the rules (i),(ii) and (iii), but not for (iv).

5. (14 points) Let k and c be real numbers. Suppose $F : \mathbb{R} \to \mathbb{R}$ and F satisfies the conditions

$$F'(x) = kF(x) \ , \qquad F(0) = c \ .$$

Prove that $F(x) = ce^{kx}$, for all x.

Solution.

See Theorem 5.4 in the text.

6. (15 points) Suppose $-\infty < a < b < \infty$ and $f\colon [a,b] \to \mathbb{R}$ is continuous. Prove that f is uniformly continuous.

Solution.

f is uniformly continuous on $[a,b]$ if either of the following conditions hold:
(i) If (u_n) and (v_n) are sequences of points from $[a,b]$ such that $\lim_n (u_n - v_n) = 0$, then $\lim_n (f(u_n) - f(v_n)) = 0$.
(ii) For every $\epsilon > 0$, there is $\delta > 0$ such that for all x and y from $[a,b]$, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon \ .$$

You could give a proof using either condition. There is a proof using (i) in the text (Theorem 3.17). I'll give a proof using (ii).

So, suppose f is not uniformly continuous. Then condition (ii) does not hold, and therefore there is an $\epsilon > 0$ for which there is no $\delta > 0$ for which the displayed implication holds. For each positive integer n, then, there are x_n and y_n from $[a,b]$ such that $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \epsilon$. Because $[a,b]$ is a closed bounded subset of $\mathbb{R}$, there is a subsequence (x_{n_k}) of (x_n) which converges to a point x in $[a,b]$. Then (using the same subsequence indices for (y_{n_k}))

$$\begin{aligned}
\lim_k y_{n_k} &= \lim_k [(y_{n_k} - x_{n_k}) + x_{n_k}] \\
&= \lim_k (y_{n_k} - x_{n_k}) + \lim_k x_{n_k} \\
&= 0 + x = x \ .
\end{aligned}$$

Then, because f is continuous at x,

$$\lim_k f(x_{n_k}) = f(x) = \lim_k f(y_{n_k}) \ .$$

Therefore

$$0 = \lim_k f(x_{n_k}) - \lim_k f(y_{n_k}) = \lim_k (f(x_{n_k}) - f(y_{n_k})) \ .$$

But this is a contradiction, because $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$ for every k. Therefore the supposition that f does not satisfy (ii) fails, and f is uniformly continuous.

7. (28 points) For each of the following, answer TRUE or FALSE. In the case that the correct answer is FALSE, give a counterexample (without proof, please) or a clear explanation. There are 4 points for each part.

(a) If E is a nonempty bounded set of real numbers, then $\sup E$ is a limit point of E.
FALSE.
For E a nonempty bounded set of real numbers, the statement fails if and only if $\sup E$ is an isolated point in E. For example, $E = \{1/n \colon n \in \mathbb{N}\}$, or $E = \{1\}$.

(b) If D is an open interval and $f \colon D \to \mathbb{R}$ is a strictly increasing differentiable function onto its image $f(D)$, then $f^{-1} \colon f(D) \to D$ is a strictly increasing differentiable function.
FALSE.
If there is an x_0 in D such that $f'(x_0) = 0$, then f^{-1} is not differentiable at $f(x_0)$. (The statement would be true if the assumption were added that $f'(x)$ is never zero.)

(c) If a, b, c and d are nonzero real numbers, then the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a + bx + cx^2 + dx^9$ has a real root.
TRUE.
It is a consequence of the Intermediate Value Theorem that a polynomial of odd degree with real coefficients has a real root.

(d) If f and g are continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and $f(x) = g(x)$ for every irrational number x, then $f(x) = g(x)$ for every x.
TRUE.
Define a function h by $h(x) = f(x) - g(x)$. Then h is a continuous function which is zero on a dense set; therefore h is identically zero. (For any x, there is a sequence (x_n) such that $h(x_n) = 0$ for every n and $\lim x_n = x$; by continuity, $h(x) = \lim h(x_n)$.)

(e) If $f : [7, 9] \to \mathbb{R}$, $f(8) = 6$ and

$$\lim_{x \to 8} \frac{\Big(f(x) - (2x - 10)\Big)}{x - 8} = 0 \quad ,$$

then f is differentiable at $x = 8$.

TRUE.

Because $f(8) = 6$, from the displayed limit we have

$$\begin{aligned} 0 &= \lim_{x \to 8} \frac{f(x) - 6 - (-6 + 2x - 10)}{x - 8} \\ 0 &= \lim_{x \to 8} \frac{f(x) - f(8) - 2(x - 8)}{x - 8} \\ 0 &= \lim_{x \to 8} \frac{f(x) - f(8)}{x - 8} - 2 \\ 2 &= \lim_{x \to 8} \frac{f(x) - f(8)}{x - 8} \quad . \end{aligned}$$

So, $f'(8) = 2$.

(f) If $f : \mathbb{R} \to \mathbb{R}$, $f(0) = 1$ and $f(x) = x$ for nonzero x, then $\lim_{x \to 0} f(x)$ does not exist.

FALSE.

$\lim_{x \to 0} f(x) = 0$. This limit is by definition independent of whether $f(0)$ is defined or what it might be.

(g) A bounded continuous function from $(0, 1)$ into the real numbers is uniformly continuous.

FALSE.

For example, define here $f(x) = \sin(1/x)$.