Combinatorial interpretation of the binomial theorem

Below k and n denote nonnegative integers satisfying $k \leq n$. If E is a finite set, then $|E|$ denotes its size, that is the number of elements contained in the set E.

Binomial coefficients. We have the definition of the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \ .$$

n-choose-k. Let $S(k,n)$ denote the collection of subsets of $\{1, 2, \dots, n\}$ which contain exactly k elements. Define $C(k,n)$ to be $|S(n,k)|$. (So, $C(k,n)$ is the number of ways to choose exactly k things from a set of n things.) In words, $C(k,n)$ is "n-choose-k". For example,

$$S(3,2) = \{ \ \{1,2\}, \{1,3\}, \{2,3\} \ \}$$

and therefore $C(3,2) = 3$.

The fundamental connection. The binomial coefficients have a combinatorial interpretation:

$$C(k,n) = \binom{n}{k} \ .$$

I'll call this the "fundamental connection". To prove it, first note that for any n we have $C(0,n) = 1$, because there is exactly one subset containing zero elements (the empty set). Also, $C(n,n) = 1$ (there is just one way to take n things from a set of n things: take them all). So the Claim is true when $k = 0$ or $k = n$.

Next, if $0 < k < n$ we make a

$$\text{CLAIM:} \quad C(k,n) = C(k-1, n-1) + C(k, n-1) \ .$$

The Claim holds for the following reason. Given $0 < k < n$, the set $S(k,n)$ can be divided into two pieces, let us call them $P = P(k,n)$ and $Q = Q(k,n)$, where P contains the elements of $S(k, n)$ which don't include n and Q contains the rest.

For example,

$$P(2,3) = \{\ \{1,2\}\ \}$$
$$Q(2,3) = \{\ \{1,3\}, \{2,3\}\ \}\ .$$

A size k subset of $\{1, 2, \ldots, n\}$ which does not contain n is a size k subset of $\{1, 2, \ldots, n-1\}$. Thus $|P(k, n)| = C(n-1, k)$.

If a size k subset of $\{1, 2, \ldots, n\}$ contains n, then after removing n we have a size $k-1$ subset of $\{1, 2, \ldots, n-1\}$; and every size $k-1$ subset of $\{1, 2, \ldots, n-1\}$ produces an element of Q, by putting in n. Thus $|Q| = C(k-1, n-1)$. Thus

$$C(n, k) = |P| + |Q| = C(n-1, k) + C(n-1, k-1)\ .$$

This proves the Claim.

Now recall that when $0 < k < n$ we have shown

$$\binom{k}{n} = \binom{k}{n-1} + \binom{k-1}{n-1}\ .$$

It is now not difficult to prove the Fundamental Connection by induction on n.

The binomial theorem.

Recall, the binomial theorem can be written

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\ .$$

For example

$$\begin{aligned}(a+b)^5 &= (a+b)(a+b)(a+b)(a+b)(a+b) \\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\ .\end{aligned}$$

If we expand $(a+b)^5$ we get $2^5 = 32$ terms, each of the form $a^{5-k}b^k$. Such a term comes from picking b from k of the 5 factors $(a+b)$ and picking a from the rest. Then we consolidate equal terms to get the formula of the binomial theorem. The number of terms of the form $a^{5-k}b^k$ is simply the number of ways to choose k of the factors as those from which b is chosen. The number of ways to do this is $\binom{5}{k}$.

For example, for $k = 1$ we have terms

$$baaaa + abaaa + aabaa + aaaba + aaaab = 5a^4b\ .$$

One more example.

$$\begin{aligned}
&(a+b)(a+b)(a+b)\\
&\qquad = a[(a+b)(a+b)] + b[(a+b)(a+b)]\\
&\qquad = a[a(a+b) + b(a+b)] + b[a(a+b) + b(a+b)]\\
&\qquad = aaa + aab + aba + abb + baa + bab + bba + bbb\\
&\qquad = aaa + (aab + aba + baa) + (abb + bab + bba) + bbb\\
&\qquad = a^3 + 3a^2b + 3ab^2 + b^3\ .
\end{aligned}$$